UNIVERSITY OF MINES AND TECHNOLOGY (UMaT)
TARKWA

FACULTY OF ENGINEERING
DEPARTMENT OF MATHEMATICAL SCIENCES

A THESIS REPORT ENTITLED

APPLICATION OF GAME THEORY IN OPTIMAL PORTFOLIO SELECTION
(A CASE STUDY, INVEST GROW SECURE FINANCIAL SERVICES)

BY
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SUBMITTED IN FULFILMENT OF THE REQUIREMENT FOR THE AWARD OF THE DEGREE OF MASTER OF PHILOSOPHY IN MATHEMATICS
(FINANCIAL MATHEMATICS)

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TARKWA, GHANA
OCTOBER, 2021

DECLARATION

I declare that this thesis is my own work. It is being submitted for the degree of Master of Philosophy in Mathematics (Financial Mathematics) in the University of Mines and Technology, Tarkwa. It has not been submitted for any degree or examination in any other University.

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(Signature of Candidate)

……….. day of ………………. (year) ……………….

DEDICATION

I dedicate this work to my grandmother and father. Both in beloved memories.

ACKNOWLEDGEMENT

I wish to thank my supervisors, Dr Henry Otoo and Dr Peter Kwesi Nyarko for their constant pressure on me, to complete this work. It is due to their continuous persistence that enable me to work extra hard for this research to meet its light of day.

A special remembrance also goes to Dr Felix Larbi Aryeh, Mercy Oparebea Mensah, Patience Mensah Larbi and Abigail Pieterson, for assisting and supporting me immensely in the completion of my Master's Degree Program.

Finally, a big thank you to all Lecturers at the Department of Mathematical Sciences for the studies, discussions and contributions during my years of study at the University.

ABSTRACT

Finance is concerned with how savings from investors are distributed to businesses via financial markets and middlemen, who then utilize them to fund their activities. After the inclusion of asset pricing’s uncertainty, it was discovered that older analysis could not explain many aspects of corporate finance.

Game theory is that part of applied mathematics that is concerned with how agents communicate with each other through their choices. It has developed a framework that has contributed to extra knowledge into many unknown phenomena, giving chance for the inclusion of asymmetric knowledge and strategic engagement.

In this study, a computational study and application of Game Theory Model to investment decisions in an optimal portfolio selection problem are considered. Emphasis was placed on an investment decision problem using data from Invest, Grow, Secure (IGS) Financial Services for the year 2018.

The decision-maker had to select at least one option from all possible options provided by IGS Financial Services in which an investment was made. The problem was to decide what action or a combination of actions to take among the various possible options with the given rates of return.

The study was successful in modeling investment options of investors as a Game Theoretic Mathematical Problem that maximizes the returns from their investments. The application of Game Theory in the financial investment strategy is also successful in offering an optimal solution as opposed to an investor's personal discretionary means. Hence, the investor is able to make better investment policies decisions based on which combinations of payoff is providing the optimal value of returns.

Furthermore, the results from the maximization of returns showed that, allocating 20 percent and 50 percent of the investor’s funds in the first quarter and second quarter respectively, yielded the maximum returns to the investor. The results from the minimization of cost however, showed that, allocating 70% of the investor’s funds in a 12 months investment policy, minimizes the most cost to the company.

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CHAPTER 1

INTRODUCTION

1.1 Background of the Study

Finance is concerned with how savings from investors are distributed to businesses via financial markets and middlemen, who then utilize them to fund their activities. At a broad level, finance is categorized into two. One is asset pricing, which concerns investor choices, and the other is corporate finance, which concerns business choices. Both forms of finance were given minimal weight in traditional neoclassical economics. This was largely associated with producing, pricing and distributing inflows and outflows, as well as associated business activities. In this regard, models expected certainty and relatively straightforward financial decisions. Significant ideas, including time worth of money and discounting have also been created using this simple approach. After the inclusion of asset pricing's uncertainty and the discovery that older analysis could not explain many aspects of corporate finance, finance developed into its own field.

Standard financial theories' failure to provide adequate explanations for observed anomalies contributes to looking for theories using new methods. This was especially true in corporate finance, in which the prevailing models were so obviously inadequate. Game theory has developed a framework that has contributed to extra knowledge into many previously unknown phenomena, giving chance for the inclusion of asymmetric knowledge and strategic engagement throughout the study. The use of Game Theory in corporate finance is discussed in fields where it has been applied the most successful to date. Afterward, its role in the pricing of assets is considered.

According to Kelly (2003), Game Theory is that part of applied mathematics that is concerned with how agents communicate with each other through their choices. This interaction, or game, is both independent - made by autonomous agents - and interdependent - the outcome is based on the agents' combination of choices. Those are strategy games where the player can readily access the information of all possible outcomes at the time of making the choice, as opposed to games of chance where the result can be decided in whole or in part by a randomized element.

Although game theory usually refers to economics and mathematics in practice, it has been commonly applied in several fields such as politics, evolutionary biology, psychology, and finance. Games as entertainment can also learn from game theory discoveries, especially as more Massively Multiplayer Online Games (MMOs) create large social and economic systems and need to balance player population choices and interactions. Tapping into human choice patterns will help developers create more realistic player-driven gameplay, regardless of whether that gameplay is of a cooperative or competitive nature.

The issue with finance is how investors' payments are made to businesses via financial markets and middlemen, who then utilize it to support their activities. The two ways finance is divided are asset pricing, which is focused with investors, and corporate finance decisions, which are focused with company decisions. In conventional neoclassical economics, neither kind of financing was given much weight. In conventional neoclassical economics, both kinds of financing were given little weight. This was largely associated with producing, pricing and distributing inflows and outflows, as well as associated business activities. In this respect, models presuppose certainty and relatively easy investment plans. Despite this simple method, key factors like the time worth of money and discounting were established.

1.2 Problem Statement

These days, there are various challenges in the area of finance. The emphasis on uncertainty and financial market processes in Keynesian macroeconomics contributes to the development of risk analysis frameworks with asset pricing. Risks were taken into account by Keynes (1936) and Hicks (1939) by the addition of a risk premium to the interest rate. Even so, there did not exist a systemic understanding behind this risk premium. Von Neumann and Morgenstern's method of option under uncertainty is an important theoretical principle that eventually leads to such a theory. The concept of expected utility, which was initially formulated for the usage in Game Theory, underpins the great majority of asset pricing theories.

Markowitz (1952; 1959) used a special case of the intended utility of von Neumann and Morgenstern to develop a portfolio choice theory, where investors only cared about the mean and variance of the portfolio payoffs chosen. Since the consumer's consumption utility is quadratic and/or asset returns are distributed multi-normally, this is a unique example of anticipated utility. Markowitz's main theory was that diversification of resources is best, and the profit that may be made is determined by the covariance of asset returns. Tobin's study on preferred cashflow, published in 1958, paved way for the establishment of the mean-variance concept as the conventional method to portfolio choice problems. Academics who came after made significant contributions to portfolio theory (Constantinides *et al.*, 1995).

Markowitz's portfolio option theory was not utilized as the basis for a theory of equilibrium until after his original contribution, notably the Capital Asset Pricing Model (CAPM). Brennan (1989), stated that the notion that those investing had the equal assumptions on the means and variances of all assets caused the delay. Sharpe (1964) and Lintner (1965) demonstrated this in an equilibrium represented as Equation (1.1).

$$Er_i = r_f + \beta_i \left(Er_M - r_f \right) \tag{1.1}$$

where

Er_i is the expected return on investment for asset i

β_i is how much risk the investment will add to a portfolio

r_f is the return on the risk-free asset

Er_M is the expected return on the market portfolio and β_i is defined as in Equation (1.2).

$$\beta_i = \operatorname{cov}\left(r_i r_M \right) / \operatorname{var}\left(r_M \right) \tag{1.2}$$

Black (1972) showed that a similar result is attained, even when their risk-free asset did not exist, provided r_f was replaced by the expected return on the portfolio.

The model specifies the Keynes (1936) and Hicks (1939) risk premium and demonstrates that it is influenced by return covariance with other assets.

Besides focusing on strong assumptions of the preferences of mean-variance and the homogeneity of investor beliefs, the CAPM was an important financial growth. More complex econometric methods were used in subsequent tests, but the results were not as encouraging. Ferson (1995) provides an analysis of such studies. The CAPM is only one of several asset-pricing models created.

Other models such as Ross's Arbitrage Pricing Theory (APT) and Lucas’s Asset-Pricing Representative Agent Model (1978) existed, however, the CAPM was the most widely used not only because it was useful for things like deriving discount rates for capital budgeting in its own right, but also because it made it simple for researchers to compensate for risk when looking at a range of issues.

The second critical area that finance discusses is the financial decisions that companies make. The decision between debt and equity, as well as the number of dividends to be paid out, were among them. Miller and Modigliani (1961) were the seminal work in this field. It is shown that the overall value of a company is independent of its debt/equity ratio with perfect markets (i.e., no frictions and symmetric information) and no taxes. Similarly, it was also shown that, the firm's worth is independent of the dividend point. It is the firm's investment decisions that are crucial in assessing its overall worth within its framework.

The significance of the theorems by Modigliani and Miller (1961) were not as described in reality. Instead, the significance of taxes and capital market inefficiencies in influencing company financial strategies should be highlighted. Incorporating interest tax deductibility, but not dividends and cost of bankruptcy, contributes to the capital structure's trade-off principle. Any debt is beneficial due to the interest deductibility tax shield, but the bankruptcy and financial hardship expenses restrict the amount to be used. In terms of dividend strategy, the introduction into the Modigliani and Miller (1961) system of the fact that revenue is paid less at the personal level than dividends result in all payouts being made by buying those shares again, rather than by paying dividends.

Capital structure trade-off theory fails to offer a sufficient description of what businesses are doing in reality. The debt tax benefit compared to the size of the estimated expense of the bankruptcy would tend to be such that companies can use more leverage than is currently observed. Miller (1977a), who attempted to address this by incorporating individual and company taxes into capital structure theory, was unsuccessful. Equity has a personal tax benefit under the Miller (1977a) model since capital gains are only taxed when they are realized, and debt has a corporate tax benefit because interest is deductible. Individuals whose income tax rates are higher than the corporate tax rate have equity, while those whose rates are lower, were having debt. This forecast contradicts what happened in the United States in the late 1980s and early 1990s, when personal tax rates were not higher than corporate tax rates. According to the Miller (1977a) model, the quantity of debt utilized by businesses should have grown dramatically, yet there was only a minor shift.

Moreover, the tax-augmented dividend theory does not provide a clear illustration of what occurs. Businesses have been paying out significant sums of their revenues as dividends for years. Efforts to solve the problem using tax-based models like the customer model were found to be unsatisfactory. They find it difficult to explain the reality that many persons in high tax rates own a large number of dividend-paying firms and pay a large amount of taxes on dividends on the margins.

Other corporate financial actions, with the exception of tax impacts and friction reductions such as transaction costs, do not produce value within the Modigliani and Miller (1961) model. While conceptual insights are given, the ideas are incompatible with what is experienced in practice. It is understandable, given their simplicity, as with the asset price models described before. Perfect information and perfect markets are particularly strong assumptions.

This study applied the Game Theory approach to give more accurate and precise results which would help both investors and investment companies in their decision making. It also focused on using a game-theoretical approach to manage investor's financial investments.

1.3 Research Objectives

The objectives of this research were to:

(i) model investment of investors as a Game Theory problem that maximizes the returns from their investments.

(ii) determine which investment options would give the investor an optimum investment yield.

(iii) determine the optimality in the investment policies.

1.4 Methods Used

Game theory has a close association with Linear Programming (LP), since every finite two-person zero-sum game can be represented as an LP, and each LP can be expressed as a game. If the problem has no saddle point, dominance fails to reduce the game, and the matrix approach fails as well, then LP is the best solution method.

The scope of the study was limited to the impact of an organization's investment management on the capital market. The research was done with help from IGS Financial

Services. Within this review, Linear Programming and Game Theory methods and concepts are explored with some related theorems and proposals. The Simplex Approach and Minimax-Maximin system to solve game issues are also covered with a brief discussion of the rectangular 2-square game.

1.5 Organization of the Thesis

Chapter 1 presents the problem statement of the theoretical models of finance and games. Chapter 2 addresses similar researches in the financial management game theoretical model. Chapter 3 describes and discusses the Game-Theoretical Optimization techniques and methods that are applied to solve the problem. Chapter 4 provides a computational study of the algorithm applied to the management of our financial investment institutions. Chapter 5 concludes with additional comments and recommendations on this thesis.

CHAPTER 2

LITERATURE REVIEW

2.1 The Problem in Finance

2.1.1 Bhattacharya’s Dividend Model

The trickiest problem in finance was what Black (1976) called "the dividend puzzle." Dividends were traditionally recompensed at a rate which was roughly half of a company's profits. Many of those dividends were earned from investors who charged large amounts of taxes on them on the margin. Furthermore, in an old study, Lintner (1956) demonstrated through a game-theoretical model that "smooth" dividends for managers are less variable than earnings. Fama and Babiak (1968) also verified this finding.

Miller and Modigliani (1961) proposed in their original article on dividends that, dividends could send out relevant information concerning prospects for a business. Nevertheless, there was little progress in understanding this problem until game-theoretical methods were applied. Bhattacharya's (1979) dividend model as a signal was among the first finance approaches to use such theoretical methods for games.

Bhattacharya (1979) believed that managers are given superior knowledge about the viability of the investment of their company. With this knowledge, they are able to signal this to the stock market when they allocate significantly large sum of the dividends. If the project turns out to be successful then such dividends are paid without any issues from earnings. When the investment is unprofitable, the organization must return to outside financing and pay the funding costs of dead weight.

As a result, the firm would only consider paying a high number of dividends if its prospects are promising. Following that, writers such as Miller and Rock (1985) and John and Williams (1985) developed game-theoretical approaches that would not need assumptions such as agreeing to a specific number of payouts, and for which the dead-weight costs needed to make the signal credible were achievable.

2.1.2 Signaling Models

One issue with dividend signaling models was that, they generally require paying dividends to convey new information. There was no point in continuing to pay for them when newer information is continuously being produced. However, the payout amount will change to reflect the new information. Smoothing is difficult to reconcile with this property of dividend signaling schemes. In a key paper, Kumar (1988) proposed a "coarse signaling" hypothesis, which was compatible with the concept that corporations have smooth dividends. Within a profit range, all industries pay the same quantity of dividends. They will change their dividend level only if they move beyond that range.

One other issue with many dividends signaling theories is that, they do not explain why firms prefer dividends to stock buybacks. Except for the manner they are priced, the two are nearly identical in most models, as both necessitate the transfer of funds from the corporation to the owners. Dividends are often taxed at high rates as ordinary income, but repurchases entail a price appreciation that is taxed at low capital gains rates. Brennan and Thakor (1996) argued that the disadvantage of repurchases is that, informed purchasers will bid for cheap stocks and avoid overpriced ones, based on the work of Ofer and Thakor (1987) and Barclay and Smith (1988). And there is an adverse issue with variety. Dividends do not suffer because they're pro-rata from this issue. In recent years some progress has been made on understanding the dividend puzzle. This is one of the game theory’s financing applications that has been a bit successful.

The capital structure trade-off principle has been a staple of the textbook for many years. While it provided a clearer understanding of businesses' decisions as compared to the original dividend models, the theory falls short since the empirical orders of magnitude of financial distress costs and interest payment shields do not appear to suit the capital structures reported. It has also benefited considerably from the application of game theory methods in this domain. (Harris and Raviv, 1991).

Signaling models were the first findings in a game-theoretic model. Ross (1977) proposed a concept in which managers choose an appropriate level of debt to indicate the company's performance to the financial markets. The fact that bankruptcy is costly, acts as a warning. These expenses are infrequently incurred by a high-debt firm with strong prospects, but they are always paid by a similarly leveraged company with terrible prospects.

Entrepreneurs utilize a company's retained portion of ownership to indicate its worth, according to Leland and Pyle (1977). The owners of high-value companies hold a significant percentage of the company to display their worth. Because of their high retention, they are unable to diversify as well as they might if symmetric information were available, making copying them unattractive to low-value firms.

Alternatively, when it is overvalued, they may have a preference for using equity. But equity is considered a poor signal. Myers (1984) used this sort of logic to develop the finance theory of "pecking order." Instead of utilizing stock to finance investment projects, it will be preferable to employ less information-sensitive sources of funding. Myers (1984) and Myers and Majluf (1984), two later articles on asymmetric knowledge, have had a significant impact. Unless executives are more informed about the firm's prospects than the stock markets, if the equity is undervalued, they would not be able to issue equity to fund investment ventures.

Dybvig and Zender (1991) emphasized that they often presume sub-optimal managerial incentives. The authors showed that the Modigliani and Miller results would hold even with asymmetric knowledge if managerial reward schemes are optimally chosen.

A second prominent trend of the capital structure literature that has used game-theoretical principles is about the costs of agencies. Jensen and Meckling (1976) referred to two forms of corporate organization issues. Another is between shareholders and bondholders, the other between shareholders and executives. The first is that investors in a leveraged business have a desire to undertake risks; the investors get the excess when returns are good, but bondholders pay the expenses when the firm defaults.

Diamond (1989) demonstrated how credibility factors when there is a long period, will boost this risk-shifting opportunity. The second dispute occurs when equity holders are unable to completely control the manager’s actions. That means managers have an opportunity to follow their interests rather than the equity holders' interests.

The company's view has also been included into a number of important studies on financial contracts written by Hart, Moore, and others. The significance of insufficient contracting possibilities in the determination of financial contracts, notably debt, was investigated using game theoretical approaches.

Hart and Moore (1989) considered a businessman wishing to raise funds to pursue a project. The project payoffs can be witnessed at each date by both the contractor and the outside investor, but they are unable to create specific contracts from such payoffs because they cannot be witnessed by third parties such as courts. Their review focuses on the issue of providing the entrepreneur with an opportunity to pay back the borrowed funds. It is demonstrated, for example, that an ideal contract is a loan agreement, and that the creditor's authority to seize the entrepreneur's assets creates possibilities to repay.

Modigliani and Miller's (1961) capital structure principle states that firms' product-market decisions are separate from their financial-market decisions. It is as simple as guaranteeing that the commodities markets are completely competitive. In an oligopolistic business with competitive disputes between enterprises on the commodities market, fiscal considerations are likely to play a significant influence. Different elements of these financial-product-market linkages were studied by Allen (1986), Brander and Lewis (1986), and Maksimovic (1986).

Allen (1986) suggested a duopoly model in which a bankrupt companies’ strategic disadvantage in picking an investment is that the bankruptcy process causes it to postpone its choice. The bankrupt firm becomes a follower in a Stackelberg investment game, rather than a contemporaneous mover in a Nash-Cournot game.

2.1.3 Oligopoly Models

In oligopoly models, Brander and Lewis (1986) and Maksimovic (1986) analyzed the role of debt as a pre-commitment system. By accepting a significant amount of debt, a company essentially pre-engages to a higher production price. The interaction between consumer and financial decisions was studied by Titman (1984) and Opler and Titman (1993). Titman (1984) investigated the impact of a higher risk of insolvency on product pricing due to the challenges of getting replacement parts and continuing a business if a company fails.

Opler and Titman (1993) looked at the link between a company's financial structure and its reputational potential to maintain high product quality. The expense of bankruptcy, which restricts debt utilization, is a significant element of the trade-off concept. The substance of some bankruptcy expenses is a key question. According to Senbet (1978), the breadth of bankruptcy expenses is restricted since firms may readily renegotiate debt conditions to avoid going bankrupt and the costs that come with it.

Game theoretic approaches were extensively employed in the research on strategic activities surrounding bankruptcy. This discovery led to Giammarino (1988), Brown (1989) and Senbet and Seward (1995). The assertion of Senbet (1978) was proven to be based on the absence of frictions. The cost of bankruptcy could be in equilibrium if there is unequal knowledge or other frictions.

Manne (1965) verbally established the idea of a competition for corporate power. The author argued that it is important to have companies managed by the most skilled and knowledgeable managers to allow the effective use of resources. The author has also suggested that the way this is done through contemporary capitalist societies is through the corporate control system. It works in many ways including tendering deals, mergers, and proxy battles.

Grossman and Hart (1980) was the paper that offered a systematic model of the takeover process and revived interest for the region. It was found out that a free-rider problem had been raised in the tender offer process. When a business offers a target to replace its management and run it more effectively, each of the target's owners has the option to refuse the offer.

The rationale for this is that they will be able to profit from the new management's advancements after that. They will auction if the bid price correctly reflects the new management's worth. As a result, tendering for the objective is not profitable for a bidding business. Furthermore, whether there are fees associated with gathering information in preparation for the offer or other bidding expenditures, the corporation will lose money. Therefore, the question of free-riders seems to preclude the possibility of takeovers. The author's solution to this problem was that the corporate charter of a company would enable acquirers after the purchase to obtain inaccessible advantages for other shareholders. We describe this process as "dilution."

Another solution offered by Shleifer and Vishny (1986) to the free-rider problem is for competitors to become stakeholders in the organization before making any official request. They will gain from the market rise in the "toehold" of shares they presently own, although they pay the premium for the shares held, they need to acquire. The data, on the other hand, disproves this assertion. Most bidders do not own any shares prior to the tender offer, according to Bradley *et al.* (1988).

A second mystery recorded by the empirical literature is the fact that bidding in contests of acquisition takes place through many large leaps, rather than several small ones. According to a survey conducted by Jennings and Mazzeo (1993), the majority of first bid premiums were more than 20% of the target's market value 10 days prior to the offer. This evidence contrasts with the English auction model's standard approach, which implies that there should be several tiny increments to the bid.

Fishman (1988) proposed an explanation that the first high premium was to discourage future competitors. Observing a tender in his model notifies the market to the target's future appeal. When the first bid is weak, the next bidder may consider the expenses to examine the target worthwhile. The second company could then bid for the target and drive the first bidder out, or compel a higher price to pay. The initial bidder will reduce the probability of this rivalry by beginning with a sufficiently high offer.

Most researches have attempted to understand the reason for their shareholders can consider the defensive measures adopted by many targets optimally. The safeguards are often designed to ensure that the customer who appreciates the product the most purchases it. Shleifer and Vishny (1986), for example, created a model in which a bidder's greenmail payment notifies other shareholders that there is no "white knight" waiting to acquire the company. This puts the company on the market, which may result in a higher price being paid than would have been the case otherwise.

2.1.4 Issues in Finance in Historic Events

Rock (1986) was the first work to provide a convincing interpretation of the phenomenon. The underpricing occurs in the author's model due to unfavorable selection. For the shares there are two classes of buyers; one is aware of the stock's true value, while the other is uninformed. The educated party can buy only when the price of the bid is below or at the true value. It means the uninformed will obtain a large allocation of overpriced stocks because they will be the only people on the market when the price of the bid is above the true value. Rock said they would be paid for the overpriced product they ended up purchasing in order to persuade the uninformed to participate. On average underpricing is one way to do this.

The hypotheses brought forward to understand underperformance, in the long run, are psychological. Miller (1977b) argued that there would be a large range of views on IPOs and that the initial price would represent the most positive view. When information is exposed over time, the most bullish investors will slowly change their expectations, and the stock price will decline.

Shiller (1990) argued that an "impresario" effect applies to the market for IPOs. Financial institutions will try to create the appearance of excess demand, which will result in underperformance at a significant cost.

Investor mood swings in the IPO market, according to Ritter (1991) and Ibbotson and Ritter (1995), and businesses exploit the "opportunity window" afforded by overpricing to issue stocks. IPOs have garnered a lot of attention in academic literature, despite the fact that they are a minor fraction of the financing activity. Perhaps the difficulty is that it's unclear how much underpricing and overpricing violate market efficiency. While game theoretical approaches were employed to explain numerous underpricing reasons, they were not utilized to justify overpricing. Alternatively, the proposed theories focused on removing investors' presumption of rational behaviour.

An area that game-theoretic models have greatly modified is intermediation. Banks and other intermediaries have traditionally been seen as ways of slashing transaction costs (Gurley and Shaw, 1960). Banking models were not quite rich.

The modeling techniques presented in Diamond and Dybvig (1983) have significantly changed the game. The study looked at a simple model where a bank offers protection against liquidity losses to depositors. Customers learn whether they will require liquidity at the intermediate or final date at the intermediate date. Long-term properties must be liquidated at some time in the middle. Customers who withdraw money are given the promised amount initially until the resources are depleted, after which they are given nothing (i.e., first come first served limitation).

Both expectations lead to two equilibriums of self-fulfillment. Everyone thinks that only those with needs at the specific date can withdraw their funds in a healthy balance, which is suitable for all depositor groups. In the fragile balance, everyone hopes that someone else will depart. Withdrawal is appropriate for early and late clients, as well as bank runs, due to the first-come, first-served rule and the expense of liquidating long-term assets. Deposit insurance, according to the authors, will eliminate the unfavorable balance. The article was also significant in terms of how liquidity needs were implemented, as well as a runner's theory, and a similar method was used to investigate other subjects.

When bank deposits are compared to what happens when shares are owned directly, Jacklin and Bhattacharya (1988) found that runs are impossible. Many depositors' models generate a warning regarding the dangerous investment. They showed that depending on the hazardous investment characteristics, either bank deposits or stocks held directly can be optimum.

Diamond (1984) proposed an assigned monitoring model in which banks could track lenders because otherwise, depositors would not be able to pay off. Bhattacharya and Thakor (1993) provided a complete account of the recent literature on banking.

Kyle (1985) proposed a framework in which there is only one risk-neutral trading platform, a collection of noisy businessmen who buy for external reasons such liquidity demands, and a risk-neutral informed trader. The market maker is in charge of setting effective prices, whilst the noisy traders are just in charge of placing orders. The experienced trader chooses an amount that would maximize his expected profit.

A risk-neutral market manager, noisy traders, and a paradigm of educated traders were introduced by Glosten and Milgrom (1985). The primary difference between this model and Kyle's is that the exchanged quantities are predetermined, and the emphasis is on establishing bid and asking prices rather than the quantity option accessible to skilled traders.

The bid demand spread is set by the stock market to account for the possibility that the trader will be aware of the real worth of the asset and will make a better estimate of it. The bid and ask prices move when orders are received, indicating the possibility that the trader will be notified. Market owners are compelled to make anticipated profits at zero, which makes the concept competitive. In addition to market microstructure, a variety of other asset-pricing subjects have been influenced by game theory. They provide models for market manipulation (Cherian and Jarrow, 1995).

Abel and Mailath (1994) proposed a risk-neutral model of investors subscribing to securities paying from the profits of a new project. It is notice that all investors will subscribe to the new securities although the expected return from all investors is negative. It did not occur unless it was common knowledge that the anticipated return of all investors was negative.

Allen *et al.* (1993) looked at the rational expectations' equilibrium in a competitive asset trading market with a finite horizon, asymmetric information, and short sales limitations. Even if every trader is aware that the asset is worthless, the item can still be sold at a profit. While each trader is aware that the asset is worthless, he adds a positive chance that another trader will give the asset a positive anticipated value in the future. Despite this, the asset is valuable. Furthermore, this could not happen if it was well known that the item was worthless. Kraus and Smith (1989) identified a model where the market is driven by the introduction of information about other people's information (not new basic knowledge).

Kraus and Smith (1989) examined a scenario in which several self-fulfilling equilibriums emerge as a result of uncertainty regarding other investors' expectations. These are known as "endogenous sunspots." These sunspots are demonstrated to be capable of generating "pseudo-bubbles," in which asset values with common knowledge are greater than in equilibrium.

Morris *et al.* (1995) looked at how decentralized and dealer marketplaces performed. While both systems do equally well in the absence of complete information, the author claims that the decentralized system performs badly when endowment uncertainty is high. The assumption is that a decentralized market needs coordination sensitive to a lack of common information, whereas dealerships require less coordination.

A more evolved literature was concerned with cascades of knowledge. Welch (1992) was a prime example. A pool of new investors must decide, whether or not to participate in an initial public offering (IPO). The developing investor is given some confidential information regarding the IPO. Assuming the first few investors pay attention to bad indications and decide not to invest, later investors would reject their private knowledge and not invest based on the (public) information suggested in other people's decisions not to participate if they got favorable signals. Despite the fact that most late-moving investors have solid knowledge, it is seldom shared with the market. While inefficiencies emerge when private information is aggregated as the behavior of the investors provides only a coarse indication of their private information. Banerjee (1992) and Bikhchandani *et al.* (1992) have studied this kind of phenomenon in more general terms.

The Devenow and Welch (1996) analysis of finance applications. The sensitive cascade statement has a major flaw in that action sets are too coarse to expose private information (Lee, 1993). This assumption is natural in some situations, such as when investors decide whether or not to commit to initial public offerings at a set offer price (although even then the requested volume may disclose details continuously). The list of potential prices (continuum) will tend to disclose prices after prices are endogenized. Nonetheless, the literature has introduced two natural explanations why knowledge cascades could occur in markets with endogenous price creation. If investors face transaction costs, based on small pieces of information (Lee, 1997), they prefer to not trade. In this scenario, market crashes may occur when large numbers of investors experience a (small) public signal that drives them into trading transaction costs, who have noticed bad news but have not acted on it.

For other reasons it does not matter if specific knowledge or gaps in priors clarify variations in beliefs. Lintner (1969), for example, derived a CAPM with heterogeneous values and – believing, as he did, that investors did not benefit from markets – did not matter the root of their differences in creeds. The distinction becomes relevant when it is believed that individuals learn from other's actions (or prices that depend on other's actions). Thus, the difference in finance started to be stressed precisely when game-theoretical and knowledge theoretical problems were brought up. Notably, "no trade" theorems, such as Milgrom and Stokey's (1982), demonstrated that differences in viewpoints based purely on knowledge disparities do not lead to a trade.

Although the difference is certainly important, it does not sustain the argument that some past convictions were not compatible (Morris, 1996). In other scenarios, there is a substantial midpoint between both the opinions financial market participants are irrational; and all religious differences are addressed by knowledge disparities.

Harrison and Kreps (1978) explored a complicated model in which traders were risk-neutral, had heterogeneous prior views about a hazardous asset's payout cycle (not explained by knowledge disparities), and were barred from short-selling that asset. Because of the opportunity benefit of being able to sell the asset in the future to any other trader with a better valuation, they noticed that the price of an asset will generally be greater than the intrinsic worth of any trader's asset (the projected discounted dividend).

Morris (1996) proposed a model in which, despite having diverse prior views, traders may learn the real dividend process over time; nonetheless, a re-sale premium develops from reflecting differences in opinion prior to learning. As a result, this model provides an explanation for the overvaluation of initial public offerings on the open market: a lack of learning opportunities indicates a larger heterogeneity of opinions, which suggests a higher price.

Harris and Raviv (1993) proposed a model in which traders dispute on the likelihood that alternative public signals are contingent on reward occurrences. A simple model that incorporates this function and predicts positive trade volume autocorrelation, as well as the link between absolute price changes and volume and a variety of other financial market data features.

Conflicts typically include the control of water supply schemes. Stakeholder activities, which might contribute to development and produce a win-win scenario, frequently result in worsening conditions for all stakeholders. Game theory may be used to define and explain parties' behaviors in relation to water resource concerns, as well as to illustrate how interactions between diverse parties who pursue their own interests rather than the system's goal result in the system's evolution. The outcomes predicted by game theory frequently differ from those indicated by optimization methods, which presume that all parties are ready to behave in the best interests of the entire system. This thesis explores game theory's applicability to water resource management and dispute resolution through a series of non-cooperative water resource games. Madani (2010) shows the complex complexity of problems with water supplies and the importance of understanding the development direction of the game when researching these issues.

Many researchers attempted studies of water dispute resolution in a game-theoretical setting. Carraro *et al*. (2005) reviewed research on game-theoretical water dispute resolution. Applications of game theory in water management literature cover a variety of water resource problems, locations, methods of solution, forms of analysis, and classifications. Many studies can be classified into more than one group. Nevertheless, attention was given to the main feature of the analysis for categorization. The authors applied the game theory for (1) water or cost/benefit allocation among users; (2) groundwater management; (3) transboundary user water allocation; (4) water quality management; and (5) other types of water resource management issues.

Many natural resource management difficulties, according to Carraro *et al.* (2005), are features of a Prisoner's Dilemma game: the dominating strategy of participants is not cooperative, and the resultant equilibrium is not Pareto-optimal. Similarly, most articles dealing with the sharing of natural resource concerns have reached the same conclusion regarding the game as the Prisoner's Dilemma. However, not all resource issues are Prisoner's Dilemmas (Sandler, 1992).

The circumstances of a question of natural resource sharing may encourage the prospect of collaboration (Taylor, 1987). Games involving water resources are not necessarily competitive (multiple users may be present, and use by one user does not preclude other users from using simultaneously). Coordination between the parties might be beneficial to both, resulting in externalities. Nonetheless, certain water management games might be thought of as anti-coordination games, in which the available resource is competed over (only one user can access the resource), sharing the resource is costly to players, and the resource is not excluded (it is difficult to stop a player from utilizing a resource if they have not paid for it). Nonetheless, certain water management games may be seen as anti-coordination games in which the accessible resource is rival (only one user can access it), sharing the resource costs players money, and the resource is not excluded (it is impossible to prohibit a player from using a resource if they have not paid for it).

The identification of the structure of water resource games is important, since the findings could be misinterpreted if the conflict modeling makes wrong assumptions. When a dispute is treated as a Prisoner's Dilemma, elements of an anti-coordination water game, for example, cannot be represented. According to Bardhan (1993), when it comes to free-riders, the literature usually jumps to the Prisoner's Dilemma.

The player may not always be able to achieve his aim on his own. When this happens (like in the Stag-Hunt game), one player collaborates while the other complies, and then differs if the other fails. In some common resource situations, the costs of defecting may be severe enough that a player will choose not to defect if the other player does (Chicken game) (Bardhan, 1993). To demonstrate that not all water resource games are prisoner's dilemmas, two non-prisoners Dilemma water resource games are shown.

Cloud computing is a newly developed concept in which a company pays when it uses a cloud provider-owned computing service. Since several clients share the resources of the cloud, they could theoretically interact with the activities of each other.

Current price and resource allocation systems are experimental (e.g., fixed pricing in Amazon EC2 / S3) and do not account for the conflict of interests that arise when several clients use the cloud at the same time. Customers may be overcharged based on the services accessible to them. These frameworks, on the other hand, do not allow the provider to make the most of its resources.

Virajith, (2003), made the first move towards modeling the dynamic interactions between client-client and client-provider in a cloud using game theory. The authors have defined a new games class named Cloud Resource Allocation Games (CRAGs). CRAGs address the problem of resource allocation in clouds using game-theoretical methods, ensuring that consumers are paid (near) optimal prices for their resource usage and that cloud resources are used within their maximum efficiency.

The authors presented the conditions for achieving different stable equilibria in CRAGs and provided algorithms that ensure near-optimal efficiency. The authors have presented results from several studies conducted using traces from Planet Lab and the Parallel Workload Archives showing that the new methods result in a performance improvement of as much as 15 percent to 88 percent relative to existing resource allocation methods such as Round-Robin.

Wireless innovations and apps are becoming ever more omnipresent in our daily lives. Wireless resources, on the one hand, are finite and fixed and, on the other, wireless technologies and devices are rising day-to-day, resulting in shortages of spectrums. Consequently, a central and fundamental stage in wireless systems is the efficient use of limited wireless resources. As demand increases, restricted wireless resource management is essential to optimally distribute the resources. In addition, the optimal distribution of available wireless resources results in knowledge being disseminated both efficiently and quickly to broad areas. Game theory has recently emerged as an effective method for optimal distribution of wireless resources, which has so-called Nash equilibrium, social optimal points.

Danda *et al*., (2009) illustrated that the Nash equilibrium does not always yield optimal results, thus the optimal point is often referred to as optimal social point. The authors first introduced the game theory and its implementation in the allocation of resources at various layers of the wireless network model's protocol stack. It also found that the static allocation of spectrum bands by government agencies such as the FCC (Federal Communications Commission) in the United States is inefficient because the licensed systems do not always use their frequency bands in full.

Users that are secondary unlicensed (cognitive radio) can classify and use the idle spectrum in an opportunistic way. For this reason, in order to provide opportunistic and optimized access to the licensed spectrum, dynamic spectrum access functionality is essential in next-generation (XG) wireless systems, which was the focus of their research. In particular, they proposed various theoretical methods for complex exposure to the spectrum of games.

Game Theory is a mathematical theory which explains the process of conflict between intelligent rational decision-makers and cooperation. The theory is especially useful in the design of wireless sensor networks (WSNs). Hai-Yan (2011) conducted a report on Game Theory's recent developments and results, its WSN applications, and offered a broad overview of this burgeoning field of study field to the community. The authors brought the standard Game Theory formulation into the WSN application domain for the first time. Game Theory roles were defined for routing protocol design, topology control, power control, and energy conservation, as well as packet forwarding, data collection, spectrum allocation, bandwidth allocation, service quality control, coverage optimization, WSN protection, and other sensor management tasks.

Then we defined three types of game theory, namely the cooperative, non-cooperative, and repeated schemes. Lastly, it identified current issues and future developments for field researchers and engineers.

Samee (2008) considers the issue of mapping activities to a digital grid to reduce power consumption and rendering duration subject to deadlines and architectural requirements. To solve this problem the author suggested a solution based on the idea of Nash Bargaining Solution from cooperative game theory. The theoretical strategy suggested for the game has been compared with other conventional techniques. The experimental results showed that the proposed strategy achieves superior performance when the deadline constraints are tight, and records competitive performance compared to the optimal solution.

Predictable distributions of security services such as police officers, canine teams, or control points are vulnerable to attacker manipulation. Recent research has applied game-theoretical approaches, including a conducted test at Los Angeles International Airport (LAX), to find optimal randomized security policies. For several related contexts, this method has promising applications including police surveillance for subway and bus systems, randomized luggage screening, and Federal Air Marshal Service (FAMS) scheduling on commercial flights. However, the existing methods scale poorly when many resources are required to be coordinated by the security policy, which is central to many of those potential applications.

Christopher (2009) created new models and algorithms that can be used in even more complicated security games The fundamental concept was to utilize a compact model of security games that allows for exponential memory and runtime savings over the most widely used methods for solving generic Stackelberg games. The authors developed significantly quicker algorithms for security games that are ubiquitous in many security areas under reward limitations. Finally, they imposed more realistic scheduling constraints while maintaining comparable performance gains. Random data and real-world examples of the FAMS and LAX issues are used in the analytical examination. The author's novel approaches can handle issues that are several orders of magnitude larger than the fastest existing algorithm.

Omar and Wessam (2011) have developed an efficient resource allocation approach in a cognitive radio network (CRN) between secondary users. The authors propose a CRN comprising a set of primary users (PUs) co-existing with secondary users (SUs) in a model for exchanging underlay spectrums. PUs uses licensed spectrum bands while SUs either choose to use unoccupied bands or coexist in the same band with PUs without disruptive primary transmissions.

The authors proposed an algorithm based on the VGC (Vickrey – Clarke – Groves) model for spectrum allocation amongst secondary transmissions in a non-cooperative game, assuming a fixed value of the bit error rate for both PUs and SUs. Its goal was to identify the most efficient and equitable way to allocate secondary broadcasts to spectrum bands with the highest data sum limit. The simulation revealed that, depending on the primary transmission power and the data rate necessary for secondary transmissions, the proposed approach maximized the sum data rate. They also proved that their proposal is about 98 percent fair using Jain's fairness index.

Corruption is a significant social and ethical problem; it needs improvements in society's beliefs, norms, and behavioral behaviors to counter it. Usually, that is a long and difficult process. Decades will pass before a society's fundamental values change. Meanwhile, corruption can be combated by adjusting incentive systems in the economy. When deep causes of the issue are thoroughly studied, a new governance structure can be set up, so that the opportunistic people do not find it lucrative to get involved in unethical practices.

Bayar (2003) presented a study that explored machine characteristics that provide a fertile environment for corruption and identified factors that promote corrupt transactions using game theoretical models. The first two models analyzed corruption as a kind of bribe-receiving trade between one bribe and the other. Through the models, the intermediaries' sector is shown to occur from agents' profit-maximizing actions. By establishing long-term, trust-based relationships with bureaucrats, this sector reduces risks arising from the fact that the two parties involved in a corrupt transaction are not perfectly familiar with each other.

Intan (2011) examined food distribution in bird broods from the cooperative game theory perspective. The goal of this study was to see if food supply data fits into the well-known collaborative game theory solution concept. The fact that the answers in the bird brood data were not immediately visible was the first issue to address. As a result, it is necessary to recreate the game using the answers supplied. A second issue is that there are numerous definitions of solution, and we want to see which one best fits our needs. The features that lead to these solutions are most fascinating since they would be most beneficial in identifying justification for the specific notion of the solution discovered in nature.

In recent years, cognitive radio technology, a groundbreaking communication model that can allow more effective use of the current wireless spectrum tools, has been attracting growing attention. Since network users need to adjust their operating conditions to the dynamic world, which may follow specific goals, conventional approaches to spectrum sharing focused on a fully cooperative, static, and centralized network environment no longer apply. Alternatively, game theory was recognized as a significant method in the research, modeling, and interpretation of the mechanism of cognitive interaction.

Wang *et al*. (2010) presented the most fundamental concepts of game theory and explained in detail how these concepts can be leveraged in the design of spectrum sharing protocols, with a focus on state-of-the-art cognitive radio networking research contributions. This also presented the work challenges and possible directions in approaches to game-theoretical modeling. The study by the author presented a detailed treatment of game theory with important applications in cognitive radio networks, which will help develop effective, self-reinforcing, distributed spectrum sharing schemes in future wireless networking.

Christian (2008) presented a theoretical game study that examined whether the changing actions of (World Trade Organization) members in the Dispute Settlement Mechanism can be explained using a theoretical game approach. To do so, the author uses a three-step process: First, an in-depth theoretical review of how the DSM operates and how the participants function in it. Second, with the result in mind from the first part, a game-theoretical model is constructed that points out various possible strategies that may be pursued in the DSM. And thirdly, the model’s effects are contrasted with the results in the first section to determine whether the model may be used to describe the specific actions of the participants of the DSM or not. The key conclusion is that certain aspects of the strategies of the member states may be clarified while the model is being developed, but when a few widely employed strategies appear entirely arbitrary from the viewpoint of the community, it is far from being a perfect explanation.

Economic dimensions of the service networks are important for real-life applications and of great importance. Current service system experiments are performed mostly using stochastic queueing networks. Game theory, a well-established method used to model the interactions between individuals, and stochastic queueing networks can be used to research the economic aspects of service systems. Choi (2010) presented a thesis using both stochastic queueing networks and game theory to research economic problems related to multi-server service systems, namely the problem of finding the optimal pricing policy, and the analysis and regulation of independently operated service providers' economic behaviours. In a service network, pricing decisions are important as they impact, apart from the benefit, consumer demand and therefore waiting times.

For a two-stage tandem queueing system with various types of customers, the model described here considered the optimal pricing scheme. The demand of every type of customer was believed to have a negative linear relationship to the service price. Analysis of authors explicitly gave the scheme for optimal pricing, which maximizes the total profits but still keeping the expected second-stage time of stay below a given level.

For the case where there is an imposition on the constraint on the total waiting time of the two stages instead of only the second stage, a further discussion has been given. Another critical and fascinating research issue concerning economic aspects of service structures is the economic conduct of the service providers in a competitive environment. The authors' study focused on the role and impact of service capacity in capturing greater market share and maximizing expected long-run profits in a multiple-server setting. They focus first on the case analysis of a common-queue service system. The problem is conceived as a strategic multiplayer game. Once the queueing mechanism is stable, equilibrium solutions are analyzed. The equilibrium service capacity is evaluated similarly and compared to the common-queue case in a separate-queue multiple-server system. The study indicates that the separate queue allocation system in the case of multiple servers provides more competitive opportunities for servers and generates higher service capacities. In particular, because there are no extreme illness economies associated with rising service capacity, a high level of reimbursement appears to support separate queue allocation.

Once John Nash earned his Ph.D. degree on non-cooperative games theory in 1950, no one could imagine that he was ahead of his time for more than several decades. Therefore, one does not guess that nearly half a century later his equilibrium would be celebrated, and the Nobel Prize will be given to Nash himself, along with other notable economists. No one today denies the importance of game theory as a science that affects other disciplines, beginning with mathematics and economy and ending with philosophy and biology. Game Theory's forefathers are widely accepted upon as being John von Neumann and Oskar Morgenstern and their Games Theory and Economic Behavior. Since then, Game Theory has grown into a vast branch of science that seeks to describe the actions of humans, social classes, businesses, and governments, as well as animals and other living organisms, collectively called teams, consistently and unambiguously.

When behaving in a specific way a player has their interest in mind. Game Theory tries to clarify the action and anticipates the best solutions possible. Dickson *et al.* (2012) submitted a paper based on the critical aspects of Game Theory and player actions in (none) alliance and (none) cooperative circumstances. The thesis focused on the art of really implementing game theories, particularly in the small business environment. Finally, the expectations are juxtaposed with real-life cases based on the knowledge presented in the previous chapters.

With this, an interview with members of small businesses was conducted, and the findings are presented in the form of observations and tips about how a small business can behave.

The importance of information dominance has been highlighted as an important ability that prospective combined resources will be able to master. It is no longer a simple future concept, but is being formally formed and integrated into doctrinal writings such as Joint Publication and Information Operations. Unfortunately, our capacity to adequately quantify its contribution to other battlefield systems remains restricted. John *et al.* (2008) investigated a model aimed at determining the boundaries of information superiority's contribution to combat outcomes, as well as the sensitivity of information superiority to variable information quality and comparing such contributions to those of other contributing elements.

In addition, an attempt is made to recognize some of the risks associated with using the supremacy of information as a force multiplier. A basic decision model was developed to answer these issues, based on the dynamics of a two-person zero-sum game. The model provides one side with varying degrees of information benefit, while the information advantage also has different degrees of information efficiency.

Additionally, several situations involving different levels of opposing side force levels were considered. Experimental design techniques were used to explore the model output space effectively while allowing for ample model replications at each design stage to provide adequate data set for analysis.

Multi-hop wireless networks are promising wireless networking techniques. The complex topology of the network and the network's selfish members make it impossible for conventional methods to model themselves. Game theory is one of the most important methods to address these issues. Most of the current works however have some limitations. A commonly accepted solution to the issue has still not been identified. Meyerson (1991) published a paper about the sharing of bandwidth in wireless networks. The author presumed that the nodes are autonomous, moral, selfish but not malicious agents in the game. Nodes are trying to send your data to the gateway in their model.

Some nodes might require others to forward their packets to connect successfully to the gateway. Nodes are selfish though and do not want to support others. Hence, some nodes may refuse to accept the requirement. In that case, by slowing down their traffic, the negative nodes will punish the others, in which case both parties will suffer. Hence, finding the equilibrium for these nodes after the negotiation cycle is non-trivial.

CHAPTER 3

METHODS USED

3.0 Overview

In this chapter, the methods and concepts of Linear Programming and Game Theory are discussed with some relevant theorems and propositions. The Simplex Method approach and the Minimax-Maximin system that addresses game issues are also covered. The chapter will also offer a brief description of the rectangular game.

The theory of games is a mathematical concept which deals with the characteristics of competitive situations such as parlor games, military operations, political campaigns, and opposing commercial enterprises' advertising and marketing techniques, among other things.

The study of human behavior is a distinct and interdisciplinary approach. Mathematics, Economics, and other Social and Behavioral Sciences are the subjects most interested in Game Theory. The great mathematician John von Neumann (1947) founded Game Theory with its principles providing a framework for formulating the structure, evaluating, and interpreting strategic scenarios.

Every finite two-person zero-sum game can be stated as Linear Programming, and every Linear Programming can be expressed as a game, game theory and linear programming have a strong relationship.

If there is no saddle point in the problem, dominance fails to minimize the game, and the matrix approach fails as well, then Linear Programming is the best solution strategy. Considering any $m \times n$ game, by turning the problem into a linear programming problem, any game with mixed strategies can be solved as such.

3.1 Linear Programming and Game Theory

Consider the Standard linear programming problem

$$\text{Maximize } Z = C^T x \tag{3.1}$$

$$\text{Subject to } Ax(\leq, =, \geq)b \tag{3.2}$$

$$x \geq 0 \tag{3.3}$$

where A is an $m \text{ x } m$ matrix and $x = \left(x_1, x_2, x_3, ..., x_n\right)$, $b = \left(b_1, b_{2,} b_{3,} ..., b_m\right)^T$ are column vectors. Consider any number of rows and columns, where $b \neq 0,$ then the system of linear equations is given in Equation (3.2). The i^{th} column of A is also denoted by $A^{(i)}$.

Objective function

The linear function $z = \sum_{j=1}^{n} C_j X_j = c_1 x_1 + c_2 x_2 + ... + c_n x_n$ that is maximized or minimized is called the objective function of the General Linear Programming Problem (GLPP).

Constraints

Constraints of a GLPP are the set of equations or inequalities. $Ax\left(\leq, =, \geq\right) = b$ is the set of constraints of the GLPP.

Solution of GLPP

An n-tuple $\left(x_1, x_2, ..., x_n\right)$ of real numbers which satisfies the constraints of a GLPP is called the solution of GLPP.

Feasible Solution

A feasible solution refers to any solution to a GLPP that confirms the non-negative restrictions of the problem. Consider a Linear Programming problem say, I of a vector $x = \left(x_1, x_2, ..., x_n\right),$ the feasible solution is the solution of the problem which satisfies the conditions $\sum_{j=1}^{n} C_j X_j \left(\leq, =, \geq\right) b_i$ for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$; $x_i \geq 0.$

Matrix form

Suppose we have found the optimal solution to (3.1). Let BV_i be the basic variable for row i of the optimal tableau. Also, define $BV = \left\{BV_1, BV_2, ..., BV_m\right\}$ to be the set of basic variables in the optimal tableau, and define the m x l vectors as,

$$X_{BV} = \begin{bmatrix} X_{BV1} \\ \vdots \\ \vdots \\ X_{BVm} \end{bmatrix}$$

Then NBV = a set of non-basic variables in the optimal tableau $X_{NBV} = (n-m)$ x l vector listing the non-basic variables. With the concept of matrix algebra, the optimal tableau n terms of BV as well as the original LP can be expressed as (3.1). Recall that $c_1, c_2, \ldots \ldots c_n$ are the objective function coefficients for the variable $x_1, x_2, \ldots, x_n$.

Here, CBV is the l x m row vector $[C_{BV1}, C_{BV1} \ldots C_{BVM}]$. Which implies that the elements of CBV are the objective function coefficients for the optimal tableau's basic variables.

$CNBV$ is the l x$(n-m)$ row vector whose elements are the coefficients of the non-basic variables (in the order of NBV).

The m x m matrix, where m is the matrix whose j^{th} column is the column for BV_j in (3.1).

a_j is the column for the variable x_j in (3.1). N is the mx$m(n-m)$ matrix whose columns are the columns for the non-basic variables (in the NBV order) in (3.1).

The mxl column vector b is the right-hand side of the constraints in (3.1).

Game

Any formal description of any situation in its strategic form is a game. (Davis, 1983).

Strategy

A player's strategy is a comprehensive assessment of all the measures that will be taken in the event of any unforeseen circumstances (Porter *et al.*, 1966).

Pay-off

The pay-off serves as a connection between the various tactics available to all participants. Consider a situation where at the end of a game, a player $p_i \, (i = 1, 2 \ldots \ldots \ldots n)$ is expected to obtain an amount v_i , then this is called the pay-off to the player p_i .

Pay-off matrix

A pay-off matrix is a table that illustrates the pay-off from player II to player I for all potential player actions (McKinsey, 1952).

Fair game

When the value of a game is equal to zero, it is considered to be a fair game.

Pure strategy

The decision to play the same row or column on every move in a game is a pure strategy for the game (Porter *et al.*, 1966).

Suppose a matrix game $A = \left(a_{ij}\right)$ has two players. If both players use their pure strategies, the outcome of each player's choice would be constant and the game is easily foreseeable. For example, if player I always chooses the i^{th} row and player II always chooses the j^{th} column, then on every play of the game player I receive $\left(a_{ij}\right)$ units from player II.

Mixed strategy

A mixed strategy is a technique of active randomization in which the player's decision is based on probability. As a specific situation, a mixed strategy can be the deterministic selection of one of the stated pure strategies.

Suppose player I decides not to play each row on each play of the game with probability 1 or 0, as was the case with pure strategies. Instead, suppose he decides to play row i with probability x_i with $i = 1, 2, \dots \dots, m,$ where more than one x_i is greater than zero and $\sum_{i}^{m} x_i = 1.$ This decision, denoted by $X = \left[x_1, x_2, \dots, x_p, \dots, x_m\right]$ is called a mixed strategy for player I (Thomas, 1969). In like manner, if player II decides to play column j with probability y_j with $j = 1, 2, \dots \dots \dots, n$ where more than one y_i is greater than zero and $\sum_{i=1}^{m} y_i = 1$ then, the decision is denoted by $Y = \left[y_1, y_2, \dots, y_p, \dots, y_m\right].$

Player

In a game, a player is an entity who makes the decisions.

Strategic form

A strategic form is a depiction of a game where the participants pick their strategy at the same time. The rewards are shown in a table, with one column for each strategy combination.

Two-person zero-sum game

The game is considered to be zero-sum if the sum of the payoffs to all players is zero for any outcome. In a two-player zero-sum game, one player's profit equals the other player's loss; hence, their interests are opposed (Harvey, 1956).

Saddle point

The saddle point is the point in a payoff matrix where the highest row minima correspond with the lowest column maxima. The payoff at the saddle point is known as the game's value, and it is clearly equal to the game's Maximin and Minimax values.

Theorem 3.1

If mixed approaches are enabled, the best match of mixed strategies according to the minimax condition proves a stable solution with V= V= V, indicating that neither player can score higher by unilaterally modifying her or his approach (Meyerson, 1991).

Theorem 3.2

In a finite matrix game, the set of optimal strategies for each player is convex and closed (Kambo, 1991).

Theorem 3.3

Let V be the value of an m x n matrix game. Then if $Y = \left[y_1, y_2, ..., y_p, ..., y_m \right]$ is an optimal strategy for player II with $y_i > 0,$ every optimal strategy x for player I must have the property.

$$\sum_{i,j=1}^{m} a_{ij} x_i = V$$

Similarly, if the optimal strategy x has $x_i > 0,$ then the optimal strategy y must be that

$$\sum_{i,j=1}^{n} a_{ij} y_i = V$$

Proposition 3.1: the set $S = \{x \mid Ax = b, x \geq 0\}$ is convex.

Proposition 3.2: $x \geq 0$ is a basic nonnegative solution of (3.2) if and only if x is a vertex of (3.1).

Proposition 3.3: If the system of equations (3.2) has a nonnegative solution, then it has a basic nonnegative solution.

Proposition 3.4: S has only a finite number of vertices (Marcus, 1969).

3.2 Simplex Method

The Simplex technique is an iterative strategy for solving standard form linear programming problems. In addition to the standard form, the constraint equations must be written as a system from which a basic feasible solution may be swiftly determined in the Simplex technique. The standard form of LP must be reduced to a variable if it is not canonical. The false variables are then removed using the two-phase approach or the Big-M method. George B. Dantzig invented the Simplex technique (1947). The Simplex approach is applicable in a variety of ways such as in economics and management science, covering financial, agricultural, industrial, transportation, and other challenges.

3.3 Minimax-Maximin Pure Strategies

Since every participant is aware the other is logical and has the same goal as them, maximizing the reward from the other, they may choose to utilize the conservative minimax criteria to choose an action. That is, player I examines each row in the payoff matrix and selects the minimum element in each row, say p_{ij} with $i = 1, 2\dots \dots \dots, m.$ Then he selects the maximum of these minimum elements, say p_{rs}.

Mathematically, $V = p_{rs} = \max\left[\min\left(p_{ij}\right)\right]$

The element p_{ij}, known as the game's maximin value, and the decision to play row r is known as the maximin pure strategy. Similarly, player II looks through each column of the reward matrix until he finds the one with the least maximum loss.

$$\text{Let, } V = p_{tu} = \min\left[\max\left(p_{ij}\right)\right]$$

Then p_{ru} is known as the game's minimax value, and the option to play column u is known as minimax pure strategy. It can be demonstrated that the minimax value reflects a lower bound on a number known as the game's value, while V represents an upper bound on the game's worth.

3.4 Rectangular 2×2 Game

This section discusses 2 x 2 game problems. First, consider a 2 x 2 game with the payoff matrix (Stanley, 1954). Let x_i be the probability player II plays row I with $i = 1, 2$ and let y_j be the probability player I plays column j with $j = 1, 2$.

Since Player II

$$\text{Player I} \quad \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \tag{3.4}$$

$$\sum_{l=1}^{2} x_i = 1 \quad \text{and} \quad \sum_{l=1}^{2} y_i = 1$$

So, we can write $x_2 = 1 - x_1$ and $y_2 = 1 - y_1$.

The game's value is intrinsically linked to the saddle point. If there is no saddle point, the steps below must be followed.

Let, the optimal strategy of player I be $\bar{y} = \begin{pmatrix} y*1 \\ y*2 \end{pmatrix}$

The optimal strategy of player II be $\bar{x} = \begin{pmatrix} x*1 \\ x*2 \end{pmatrix}$

$$y*1 = \frac{P_{22} - P_{21}}{P_{11} + P_{22} - P_{12} - P_{21}} \tag{3.5}$$

$$y*2 = 1 - y*1 \tag{3.6}$$

$$x*1 = \frac{P_{22} - P_{21}}{P_{11} + P_{22} - P_{12} - P_{21}} \tag{3.7}$$

$$x*2 = 1 - x*1 \tag{3.8}$$

For player I and player II, these will be the best minimax tactics. Finally, the game's worth may be summed up as follows:

$$V = (y*1)(x*1)P_{11} + (y*1)(1-x*1)P_{12} + (1-y*1)(x*1)P_{21} + (1-y*1)(1-x*1)P_{21} \tag{3.9}$$

3.5 Solving Game Problems, Reducing to LP

Here, generalized $m \times n$ game problems as discussed by converting them into Linear Programming. The LP technique is used to determine the approaches of the two players.
In a wide range of applications, simple solutions to a system of linear equations must be determined. It is usually easier to find the extreme points in many linear programming problems, such as degenerate and cycle difficulties, by utilizing the typical simplex approach (Beale, 1955).

3.6 An *mxn* Game

By turning the problem into a linear programming problem, any game with mixed strategies can be solved. Let the value of the game be v. If player I acts to maximize and player II acts to minimize, then the result has an objective function that corresponds to the value of the game. But, if player I acts to minimize whiles player II acts to maximize, then the LP is converted into the reciprocal of the original LP, and the inverse of the value of the game is obtained. In this situation, the objective function also changes.

First, consider, the optimal mixed strategy for player II, The expected payoff for player II $=\sum_{i}^{m}\sum_{j}^{n} p_{ij} y_j x_i$ and the player II strategy $(x_1, x_2, ..., x_m)$ is optimal if $\sum_{i}^{m}\sum_{j}^{n} p_{ij} y_j x_i \leq v$ for all opposing strategies, i.e., player I strategy $(y_1, y_2, ..., y_n)$.

By maximizing both payoffs for player II and player I, the new equations for these players become;

Player II: Maximize $\frac{1}{v} = x_1 + x_2 + ... + x_m$ (3.10)

Subject to

$$p_{11}x_1 + p_{12}x_2 + ... + p_{1n}x_n \leq 1$$

$$p_{21}x_1 + p_{22}x_2 + ... + p_{2n}x_n \leq 1 \tag{3.11}$$

.

.

.

$$p_{m1}x_1 + p_{m2}x_2 + ... + p_{mn}x_n \leq 1$$

$x_1 + x_2 + ... + x_n = 1$ and $x_i \geq 0,$ for $j = 1, 2, ..., n$

Player I: Maximize $\frac{1}{v} = y_1 + y_2 + ... + y_m$ (3.12)

Subject to

$$p_{11}y_1 + p_{21}y_2 + ... + p_{m1}y_m \geq 1$$

$$p_{12}y_1 + p_{22}y_2 + ... + p_{m2}y_m \geq 1 \tag{3.13}$$

$$p_{1n}x_1 + p_{2n}y_2 + ... + p_{mn}y_m \geq 1$$

$$y_1 + y_2 + ... + y_m = 1$$

And $y_j \geq 0,$ for $i = 1, 2, ..., \mathrm{m}$

Equation (3.10) and equation (3.12) can be solved using an effective LP method such as the normal Big M Simplex Method or the Primal-Dual Simplex Method. The regular Simplex Process is also improved in both speed and accuracy by the integration of computer software. This method can solve the problems with regards to popularity and in the considered real-life problem, the implementation of the above procedure showed the advantages.

Algorithm

Here is the algorithm of the game by Minimax-Maximin, 2×2 strategies, and the modified matrix of the game problems.

Step 1: When the pay-off matrix is $2x2$ then, find the value of the game.

Substep (i): Find the maximum element in each row of the payout matrix in equation (3.4).

Substep (ii): Find the smallest entry in each column of the payout matrix in equation (3.4).

Substep (iii): If they match, the game's value is V= Maximin element = Minimax element. Then Stop. If they are unable to get such value, move to Substep (iv).

Substep (iv): Identify the mixed strategies for Player I using (3.5) and (3.6).

Substep (v): Identify the mixed strategies for Player II using (3.7) and (3.8).

Substep (vi): Finally, the solution of the game is obtained by (3.9).

If not, go to Step (2) for m, n >2.

Step 2: Find the least element in each row of the reduced payoff matrix, then the maximum element of these minimal elements.

Step 3: Find the smallest element of each maximal element in each column of the reduced payoff matrix.

Step 4: If the Maximin is smaller than zero for player I, determine k, which is equal to one plus the absolute value of Maximin.

Step 5: If the Minimax is smaller than zero for player II, determine k, which is equal to one plus the absolute value of Minimax.

Step 6: If Maximin and Minimax both are greater than zero then $k \geq 0$.

Step 7: Finally, add k to each payout element of the current payoff matrix to get the updated payoff matrix.

Step 8: Then, using the technique below, discover the mixed tactics with the game value of the two players.

Algorithm for player I and player II

For $m \text{ x } m$ game problems, a computational technique based on the simplex method is provided in terms of certain stages for determining their strategies using the game value from the updated matrix.

Step 1: First, identify the payoff matrix for player II and player I and the value of k.

Step 2: This gives equations (3.11) and (3.13) for player II and player I respectively.

Step 3: Entries for player II is taken from the equation (3.11).

Step 4: State the types of constraints. If all are of "≤" type move to step (6).

Step 5: Then, follow the following sub-step.

Sub-step (i): Express the problem in standard form.

Sub-step (ii): Start with an initial basic feasible solution in canonical form and set up the initial tableau.

Step 2: This gives equations (3.11) and (3.13) for player II and player I respectively.

Step 3: Input for player II is taken from the equation (3.11).

Step 4: Define the types of constraints. If all are of $"\leq"$ type go to step (6).

Step 5: Then, follow the following sub-step.

Sub-step (i): Express the problem in standard form.

Sub-step (ii): Set up the first table with an initial basic feasible solution in canonical form.

Sub-step (iii): Use the inner product rule to find the relative profit factors $\overline{C_j}$ as follows $\overline{C_j} = C_j - Z_j$ (inner product of C_B and the column corresponding to x_j in the canonical system).

Sub-step (iv): If all $\overline{C_j} \leq 0,$ the current basic feasible solution is optimal and stop.

Otherwise, select the non-basic variable with the most positive C_j to enter the basis.

Sub-step (v): Select the pivot operation to get the tableau and basic feasible solution.

Sub-step (vi): Move to Sub-step (iii).

Step (6): Introduce slack and excess variables to begin expressing the problem in standard form. Then, if necessary, include artificial variables to express the problem in canonical form, and create the initial basic viable solution. Move to Sub-step (iii).

Step (7): When any C_j which corresponds to a non-basic variable is zero, the problem has an alternative solution, take this column and move to Sub-step (v).

Step (8): Finally, we find all the strategies for player II are in corresponding their right-hand side (RHS), and strategies of player I are in corresponding the $\overline{C_j} = C_j - Z_j$ of the slack variables.

Step (9): Calculate the value of each possible solution's objective functions.

3.7 Sensitivity Analysis (What-if Analysis)

A sensitivity analysis examines how alternative values of an independent variable impact a certain dependent variable under a set of assumptions. To put it another way, sensitivity analysis looks at how different sources of uncertainty in a mathematical model impact the model's overall uncertainty. This technique is used inside specific parameters that are reliant on one or more input factors (Anon., 2021a).

One or more input variables within the defined parameters, such as the impact of interest rate changes on the price of a bond, will decide its implementation. Sensitivity analysis may help any activity or system. Sensitivity analysis may be used for a variety of purposes, from planning a family trip to making business decisions. The basic premise of sensitivity analysis is that the model parameters will be changed and the behavior will be monitored.

Sensitivity analysis is among the methods that can give decision-makers with more than a solution to a problem. It provides you with a thorough awareness of the flaws in the model being considered. Finally, the decision-maker has a clear idea of how susceptible the optimal solution he picks is to changes in the input values of one or more parameters.

3.7.1 Local Sensitivity Analysis

The derivatives are used in the local sensitivity analysis (numerical or analytical). The word "local" refers to the derivatives being computed at a specific location. This method works well for basic cost functions, but it is ineffective for more complicated models, such as those with discontinuities and no derivatives.

Local sensitivity analysis is a one-at-a-time (OAT) approach that looks at the effect of one parameter on the cost function while keeping the other parameters constant.

3.7.2 Global Sensitivity Analysis

Global sensitivity analysis, which is usually carried out using Monte Carlo methods, is the second approach to sensitivity analysis. This approach uses a huge number of samples to explore the design space (Anon., 2021b).

CHAPTER 4

DATA COLLECTION AND ANALYSIS

4.0 Overview

In this chapter, a theoretical analysis of Game Theory is applied in an Optimal Portfolio Selection Problem for investment decisions. Emphasis is placed on the investment decision problem, which is modeled as the problem of game theory. IGS Financial Services Limited data for the year 2018 is examined.

4.1 Data Collection and Analysis

The Treasury Department of IGS Financial Services Limited, Tarkwa, Ghana, contributed the data for this thesis. IGS Financial Services Limited is a financial services firm incorporated under the Companies Act of 1960 (Act 179), registered as a Pensions Fund Manager with the National Pensions Regulatory Authority, and licensed by the Securities and Exchange Commission to conduct investment research and publication, as well as portfolio management of securities on behalf of clients for investment purposes. They usually provide a variety of financial investment services as well as consulting on the types of investments that customers should undertake.

The following set of data in Table 4.1 below is the 2018 financial year data from the treasury department of IGS Financial Services Limited.

Table 4.1 Investment Options for the 2018 Financial Year

Month	Bonds, Stocks, Deposits, Mutual Funds and Treasury Bills (%)			
	3 months	6 months	12 months	24 months
January	13.41	14.33	15.00	18.00
February	13.37	14.19	15.00	18.00
March	13.36	13.93	14.50	18.00
April	13.39	14.64	15.00	18.00

May	14.39	14.96	15.00	19.50
June	13.41	15.15	15.00	19.50
July	14.41	14.13	17.83	19.50
August	14.53	14.81	15.50	19.70
September	14.80	14.65	17.90	19.70
October	14.61	13.44	15.00	19.70
November	14.48	14.82	15.30	19.50
December	14.56	15.00	15.50	19.50

Table 4.1 shows the various IGS investment options available to a client for the 2018 financial year. It states in percentages, the Bonds, Stocks, Deposits, Mutual Funds, and Treasury Bills for 3 months, 6 months, 12 months, and 24 months.

Table 4.2 Investment Options in the 4 Quarters of the Year 2018

	Player I				
Player II	**Month**	**Bonds, Stocks, Deposits, Mutual Funds and Treasury Bills (%)**			
		3 months	**6 months**	**12 months**	**24 months**
	Jan-Mar	13.38	14.15	14.83	18.00
	Apr-Jun	13.73	14.92	15.00	19.00
	July-Sep	14.58	14.53	17.08	19.63
	Oct-Dec	14.55	14.42	15.27	19.57

Table 4.2 shows the various IGS investment options available to a client for the 2018 financial year in a 3-month interval. The mean of the first quarter is calculated for all periods of Bonds, Stocks, Deposits, Mutual Funds, and Treasury Bills. The same applies to the second quarter, third quarter, and fourth quarter.

The decision-maker must choose at least one option from among all those available to invest in. The issue is deciding which action (or combination of actions) to perform from the many alternatives available at the specified rates of return. When comparing the above situation to a normal game theory problem, the investor is essentially playing a game against the Investment Company.

To solve the above problem, the game is first checked to see if it has a Saddle point.

Thus, at the Saddle point, $Min\ \{Column\ Maximum\} = Max\ \{Row\ Minimum\}$

Table 4.3 Finding Column Maximum and Row Minimum

<table>
<tr><td></td><td colspan="6">Player I</td></tr>
<tr><td rowspan="7">Player II</td><td rowspan="2">Month</td><td colspan="4">Bonds, Stocks, Deposits, Mutual Funds and Treasury Bills (%)</td><td rowspan="2">Row Min</td></tr>
<tr><td>3 months</td><td>6 months</td><td>12 months</td><td>24 months</td></tr>
<tr><td>Jan-Mar</td><td>13.38</td><td>14.15</td><td>14.83</td><td>18.00</td><td>13.38</td></tr>
<tr><td>Apr-Jun</td><td>13.73</td><td>14.92</td><td>15.00</td><td>19.00</td><td>13.73</td></tr>
<tr><td>July-Sep</td><td>14.58</td><td>14.53</td><td>17.08</td><td>19.63</td><td>14.53</td></tr>
<tr><td>Oct-Dec</td><td>14.55</td><td>14.42</td><td>15.27</td><td>19.57</td><td>14.42</td></tr>
<tr><td>Column Max</td><td>14.58</td><td>14.92</td><td>17.08</td><td>19.63</td><td></td></tr>
</table>

From Table 4.3, the maximum values of the columns are shown as 14.58, 14.92, 17.08 and 19.63. This implies that, the $Min\ \{Column\ Maximum\} = 14.58$. Likewise, the minimum values of the rows are shown as 13.38, 13.73, 14.53, and 14.42. This also implies that, the $Max\ \{Row\ Minimum\} = 14.53$.

Since $Min\ \{Column\ Maximum\} \neq Max\ \{Row\ Minimum\}$ the game has no Saddle point.

From the Investment Company's point of view, the opponent is a maximizing player, while from the investor's point of view, the opponent is a minimizing player. Thus, the problem is solved as a Linear Programming Problem. Let x_i, where $i = 1, 2, 3 \text{ and } 4$ be the probabilities with which Investment Company plays their strategies and y_j, where $j = 1, 2, 3$ and 4 with which the investor plays his strategies. Then the inequalities of the investor can be expressed in compact form as:

$$\text{Maximize Z=}\sum_{j=1}^{n} Y_j \qquad (4.1)$$

The objective function here applies that, the investor makes as much profit from the initial deposit by combining the best options available within specific month intervals. This is represented as Equation (4.2).

$$\text{Subject to } \sum_{i,j=1}^{n} R_{ij} y_j \qquad (4.2)$$

where the investor has to choose at least one of the investment options available from each month interval $\forall\, y_j \geq 0$.

This can be expanded mathematically as Maximize $Z = Y_1 + Y_2 + Y_3 + Y_4$

Subject to

$$13.38Y_1 + 14.15Y_2 + 14.83Y_3 + 18.00Y_4 \leq 1$$

$$13.73Y_1 + 14.92Y_2 + 15.00Y_3 + 19.00Y_4 \leq 1$$

$$14.58Y_1 + 14.53Y_2 + 17.08Y_3 + 19.63Y_4 \leq 1$$

$$14.55Y_1 + 14.42Y_2 + 15.27Y_3 + 19.57Y_4 \leq 1$$

$$Y_1 \geq 0,\ Y_2 \geq 0,\ Y_3 \geq 0,\ Y_4 \geq 0$$

where the total interest on the initial income will not exceed 100%.

This maximization problem can be worked out by the simplex method. Since the inequalities are of the less than type, to change them into equalities, slack variables are added to each of the inequalities to obtain the equations for the simplex method. Taking s_1, s_2, s_3 and s_4 as the slack variables, the problem can be stated as Equation (4.3):

$$\text{Maximize: } Z = Y_1 + Y_2 + Y_3 + Y_4 + 0\, s_1 + 0\, s_2 + 0\, s_3 + 0\, s_4 \qquad (4.3)$$

which are subject to a system of equations in Equation (4.4) below:

$$\left.\begin{aligned} 13.38Y_1 + 14.15Y_2 + 14.83Y_3 + 18.00Y_4 + s_1 + 0s_2 + 0s_3 + 0s_4 &= 1 \\ 13.73Y_1 + 14.92Y_2 + 15.00Y_3 + 19.00Y_4 + 0s_1 + s_2 + 0s_3 + 0s_4 &= 1 \\ 14.58Y_1 + 14.53Y_2 + 17.08Y_3 + 19.63Y_4 + 0s_1 + 0s_2 + s_3 + 0s_4 &= 1 \\ 14.55Y_1 + 14.42Y_2 + 15.27Y_3 + 19.57Y_4 + 0s_1 + 0s_2 + 0s_3 + s_4 &= 1 \end{aligned}\right\} \quad (4.4)$$

$$Y_1 \geq 0,\ Y_2 \geq 0,\ Y_3 \geq 0,\ Y_4 \geq 0,\ s_1 \geq 0,\ s_2 \geq 0,\ s_3 \geq 0,\ s_4 \geq 0$$

From the above, the first simplex tableau for the maximizing the profit of the investor can be set up as below.

Table 4.4 The First Simplex Tableau for Maximization

c_j	Basic variable	Value of the Basic variable	1	1	1	1	0	0	0	0
			Y_1	Y_2	Y_3	Y_4	s_1	s_2	s_3	s_4
0	s_1	1	13.38	14.15	14.83	18.00	1	0	0	0
0	s_2	1	13.73	14.92	15.00	19.00	0	1	0	0
0	s_3	1	14.58	14.53	17.08	19.63	0	0	1	0
0	s_4	1	14.55	14.42	15.27	19.57	0	0	0	1
	z_j	0	0	0	0	0	0	0	0	0
	$c_j - z_j$									

For maximum output, the following set of equations is considered as Equation (4.5)

$$\left.\begin{aligned} z-13.38Y_1-14.15Y_2-14.83Y_3-18.00Y_4-s_1&=0\\ z-13.73Y_1-14.92Y_2-15.00Y_3-19.00Y_4-s_2&=0\\ z-14.58Y_1-14.53Y_2-17.08Y_3-19.63Y_4-s_3&=0\\ z-14.55Y_1-14.42Y_2-15.27Y_3-19.57Y_4-s_4&=0 \end{aligned}\right\} \qquad (4.5)$$

By comparison, to obtained the maximum value of z, the intersection of column Y_4 and row s_3 is considered, since it is the entry with the highest investment value, therefore row s_3 becomes the pivot row, column Y_4 becomes the pivot column and the entry, 19.63 becomes the pivot element.

Similarly, the compact form (minimization) for the Investment Company is represented as Equation (4.6)

$$\text{Minimize Z=}\sum_{i=1}^{n} X_i \qquad (4.6)$$

The objective function here applies that, the Investment Company minimizes as much cost from the initial deposit by combining the best options available within specific month intervals. This is represented as Equation (4.7)

$$\text{Subject to } \sum_{i,j=1}^{n} R_{ij} x_i \qquad (4.7)$$

where the Investment Company has to provide at least one of the investment options available from each month interval $\forall\ x_i \geq 0$

This can also be expanded mathematically as Minimize $Z = X_1 + X_2 + X_3 + X_4$

Subject to $13.38X_1 + 13.73X_2 + 14.58X_3 + 14.55X_4 \geq 1$

$$14.15X_1 + 14.92X_2 + 14.53X_3 + 14.42X_4 \geq 1$$

$$14.83X_1 + 15.00X_2 + 17.08X_3 + 15.27X_4 \geq 1$$

$$18.00X_1 + 19.00X_2 + 19.63X_3 + 19.57X_4 \geq 1$$

$$X_1 \geq 0,\ X_2 \geq 0,\ X_3 \geq 0,\ X_4 \geq 0$$

where the total interest on the initial cost will not exceed 100%.

This minimization problem can be worked out by the simplex method. Since the inequalities are of the greater than type, to change them into equalities, slack variables are deducted from each of the inequalities to obtain the equations for the simplex method. Taking s_1, s_2, s_3 and s_4 as the slack variables, the problem can be stated as Equation (4.8):

$$\text{Minimize: } Z = X_1 + X_2 + X_3 + X_4 - 0\,s_1 - 0\,s_2 - 0\,s_3 - 0\,s_4 \quad (4.8)$$

which are subject to a system of equations in Equation (4.9) below:

$$\left.\begin{aligned}
&13.38X_1 + 14.15X_2 + 14.83X_3 + 18.00X_4 - s_1 - 0s_2 - 0s_3 - 0s_4 = 1 \\
&13.73X_1 + 14.92X_2 + 15.00X_3 + 19.00X_4 - 0s_1 - s_2 - 0s_3 - 0s_4 = 1 \\
&14.58X_1 + 14.53X_2 + 17.08X_3 + 19.63X_4 - 0s_1 - 0s_2 - s_3 - 0s_4 = 1 \\
&14.55X_1 + 14.42X_2 + 15.27X_3 + 19.57X_4 - 0s_1 - 0s_2 - 0s_3 - s_4 = 1
\end{aligned}\right\} \quad (4.9)$$

$$X_1 \geq 0,\ X_2 \geq 0,\ X_3 \geq 0,\ X_4 \geq 0,\ s_1 \geq 0,\ s_2 \geq 0,\ s_3 \geq 0,\ s_4 \geq 0$$

From the above, the first simplex tableau for the minimization of cost to the investor company can be set up as below.

Table 4.5 The First Simplex Tableau for Minimization

c_j	Basic variable	Value of the Basic variable	1	1	1	1	0	0	0	0
			X_1	X_2	X_3	X_4	s_1	s_2	s_3	s_4
0	s_1	1	13.38	14.15	14.83	18.00	1	0	0	0
0	s_2	1	13.73	14.92	15.00	19.00	0	1	0	0
0	s_3	1	14.58	14.53	17.08	19.63	0	0	1	0
0	s_4	1	14.55	14.42	15.27	19.57	0	0	0	1
	z_j	0	0	0	0	0	0	0	0	0
	$c_j - z_j$									

Therefore, the dual value problem for both maximization and minimization are shown in Table 4.6 below.

Table 4.6 The Maximization and Minimization Problems of X and Y

Original Problem						
Maximize	Y_1	Y_2	Y_3	Y_4		
Constraint 1	13.38	14.15	14.83	18.00	≤	1
Constraint 2	13.73	14.92	15.00	19.00	≤	1
Constraint 3	14.58	14.53	17.08	19.63	≤	1
Constraint 4	14.55	14.42	15.28	19.57	≤	1
Dual Problem						
Minimize	Constraint 1	Constraint 2	Constraint 3	Constraint 4		
X_1	13.38	13.73	14.58	14.55	≥	1

X_2	14.15	14.92	14.53	14.42	$\geq$	1
X_3	14.83	15.00	17.08	15.28	$\geq$	1
X_4	18.00	19.00	19.63	19.57	$\geq$	1

4.2 Results

4.2.1 Maximization of Investor's Payoff

For maximizing the payoffs of the investor, the various feasible solutions from the iterations generated by the Production Operations Management - Quantitative Method (POM-QM) Optimization Software are shown in Table 4.7.

Table 4.7 Values Showing the Various Iterations for the Maximization Problem

c_j	Basic variable	Value of the Basic variable	1	1	1	1	0	0	0	0
			Y_1	Y_2	Y_3	Y_4	s_1	s_2	s_3	s_4
Iteration 1										
0	s_1	1	13.38	14.15	14.83	18.00	1	0	0	0
0	s_2	1	13.73	14.92	15.00	19.00	0	1	0	0
0	s_3	1	14.58	14.53	17.08	19.63	0	0	1	0
0	s_4	1	14.55	14.42	15.27	19.57	0	0	0	1
	z_j	0	0	0	0	0	0	0	0	0
	$c_j - z_j$		1	1	1	1	0	0	0	0

Iteration 2										
0	s_1	0.082	0	0.816	-0.844	-0.014	1	0	-0.918	0
0	s_2	0.058	0	1.237	-1.084	0.514	0	1	-0.942	0
1	Y_1	0.069	1	0.997	1.172	1.346	0	0	0.069	0
0	s_4	0.002	0	-0.080	-1.765	-0.020	0	0	-0.998	1
	z_j	0.069	1	1.000	1.170	1.350	0	0	0.070	0
	$c_j - z_j$		0	0.003	-0.172	-0.346	0	0	-0.069	0
Iteration 3										
0	s_1	0.044	0	0	-0.129	-0.354	1	-0.660	-0.297	0
1	Y_2	0.047	0	1	-0.877	0.416	0	0.808	-0.761	0
1	Y_1	0.022	1	0	2.045	0.932	0	-0.806	0.827	0
0	s_4	0.006	0	0	-1.835	0.014	0	0.065	-1.059	1
	z_j	0.069	1	1	1.170	1.350	0	0.000	0.070	0
	$c_j - z_j$		0	0	-0.169	-0.348	0	-0.003	-0.066	0

The various range of values associated with the options of investments and their reduced cost are shown in Table 4.8 below.

Table 4.8 Values Showing the Ranges of Investment Maximization Options

Variable	Value	Reduced Cost	Original Val	Lower Bound	Upper Bound
Y_1	.02	0	1	.92	1.0
Y_2	.05	0	1	1	1.09
Y_3	0	.17	1	-Infinity	1.17
Y_4	0	.35	1	-Infinity	1.35
Constraint	**Dual Value**	**Slack/Surplus**	**Original Val**	**Lower Bound**	**Upper Bound**
Constraint 1	0	.04	1	.96	Infinity
Constraint 2	0	0	1	.94	1.03
Constraint 3	.07	0	1	.97	1.01
Constraint 4	0	.01	1	1	Infinity

This implies that for each value of Y_1, Y_2, Y_3 and Y_4 within the lower and the upper bound, the optimal solution will remain unchanged, regardless of whatever combination of choices to maximize profit or reduce cost. The solution lists are also shown in Table 4.9.

Table 4.9 Various Feasible Solutions of the Investment Maximization Options

Variable	Status	Value
Y_1	Basic	.02
Y_2	Basic	.05
Y_3	NON-Basic	0
Y_4	NON-Basic	0
s_1	Basic	.04

S_2	NON-Basic	0
S_3	NON-Basic	0
S_4	Basic	.01
Optimal Value (Z)		**6.5**

Similarly, a series of combinations for the various options will still produce the same optimal solution of 6.5 within the boundaries.

A further test for this approach of portfolio selection for investment options is applied using Sensitivity Analysis. With this, the first column and second row (Scenario 1), and the fourth column and third row (Scenario 2) of the 4×4 matrix were chosen at random, to sensitize. A 5% increment was added in each scenario and the outcome was determined. This was done to vary the values of the data provided and to check how much effect will occur if there was a 5% change in the originally expected outcome. The details for the new matrix for Scenario 1 are shown in table 4.10.

Table 4.10 Values for a 5% Increment for Scenario 1

Month	**Bonds, Stocks, Deposits, Mutual Funds and Treasury Bills (%)**				**Row Min**
	3 months	**6 months**	**12 months**	**24 months**	
Jan-Mar	14.05	14.15	14.83	18.00	**14.05**
Apr-Jun	14.42	15.67	15.75	19.95	**14.42**
July-Sep	15.31	14.53	17.08	19.63	**14.53**
Oct-Dec	15.28	14.42	15.27	19.57	**14.42**
Column Max	**15.31**	**15.67**	**17.08**	**19.95**	

Since $Min\ \{Column\ Maximum\} \neq Max\ \{Row\ Minimum\}$ the game has no Saddle point.

Therefore, the values showing the iterations generated by the POM-QM optimization software are shown in table 4.11.

Table 4.11 Values Showing the Various Iterations for Scenario 1

c_j	Basic Variables	Quantity	$1Y_1$	$1Y_2$	$1Y_3$	$1Y_4$	s_1	s_2	s_3	s_4
Iteration 1										
0	s_1	1	14.05	14.15	14.83	18.00	1	0	0	0
0	s_2	1	14.42	15.67	15.75	19.95	0	1	0	0
0	s_3	1	15.31	14.53	17.08	19.63	0	0	1	0
0	s_4	1	15.28	14.42	15.27	19.57	0	0	0	1
	z_j	0	0	0	0	0	0	0	0	0
	$c_j - z_j$		1	1	1	1	0	0	0	0
Iteration 2										
0	s_1	0.082	0	0.816	-0.844	-0.015	1	0	-0.918	0
0	s_2	0.058	0	1.985	-0.337	1.461	0	1	-0.942	0
1	Y_1	0.065	1	0.949	1.116	1.282	0	0	0.065	0
0	s_4	0.002	0	-0.082	-1.777	-0.022	0	0	-0.998	1
	z_j	0.065	1	0.950	1.120	1.280	0	0	0.070	0
	$c_j - z_j$		0	0.051	-0.116	-0.282	0	0	-0.065	0
Iteration 3										
0	s_1	0.058	0	0	-0.706	-0.615	1	-0.411	-0.531	0
1	Y_2	0.029	0	1	-0.170	0.736	0	0.504	-0.475	0
1	Y_1	0.038	1	0	1.277	0.584	0	-0.478	0.516	0
0	s_4	0.004	0	0	-1.790	0.039	0	0.041	-1.037	1
	z_j	0.067	1	1	1.110	1.320	0	0.030	0.040	0

	$c_j - z_j$		0	0	-0.107	-0.320	0	-0.026	-0.041	0

Given the results from the above iterations, the various feasible solutions for Scenario 1 are shown in table 4.12.

Table 4.12 Values Showing the Various Feasible Solutions for Scenario 1

Variable	**Status**	**Value**
Y_1	Basic	0.04
Y_2	Basic	0.03
Y_3	NON-Basic	0
Y_4	NON-Basic	0
s_1	Basic	0.06
s_2	NON-Basic	0
s_3	NON-Basic	0
s_4	Basic	0
Optimal Value (Z)		**6.7**

The details for the new matrix for Scenario 2 are also as shown in table 4.13 below.

Table 4.13 Values for a 5% Increment for Scenario 2

Month	**Bonds, Stocks, Deposits, Mutual Funds and Treasury Bills (%)**				**Row Min**
	3 months	**6 months**	**12 months**	**24 months**	
Jan-Mar	13.38	14.15	14.83	18.90	**13.38**
Apr-Jun	13.73	14.92	15.00	19.95	**13.73**
July-Sep	15.31	15.26	17.93	20.61	**15.26**
Oct-Dec	14.55	14.42	15.27	20.55	**14.42**
Column Max	**15.31**	**15.26**	**17.93**	**20.61**	

Since $Min\ \{Column\ Maximum\} = Max\ \{Row\ Minimum\}$ the game has a Saddle point.

Therefore, the investor can maximize their profit the most within July-September, whiles IGS Financial Services Limited can also minimize their cost the most by intervals of 6 months.

4.2.2 Minimization of Investment Company's cost

For minimization of costs of IGS Financial Services, the various feasible solutions from the iterations generated by the Production Operations Management - Quantitative Method (POM-QM) Optimization Software are shown in Table 4.14 below.

Table 4.14 Values Showing the Various Iterations for the Minimization Problem

c_j	Basic Variables	Value	1	1	1	1	0	0	0	0	0	0	0	0
			X_1	X_2	X_3	X_4	a_1	δ_1	a_2	δ_2	a_3	δ_3	a_4	δ_4
Phase 1 - Iteration 1														
1	a_1	1.00	13.38	13.73	14.58	14.55	1.00	-1.00	0.00	0.00	0.00	0.00	0.00	0.00
1	a_2	1.00	14.15	14.92	14.53	14.42	0.00	0.00	1.00	-1.00	0.00	0.00	0.00	0.00
1	a_3	1.00	14.83	15.00	17.08	15.28	0.00	0.00	0.00	0.00	1.00	-1.00	0.00	0.00
1	a_4	1.00	18.00	19.00	19.63	19.57	0.00	0.00	0.00	0.00	0.00	0.00	1.00	-1.00
	z_j	4.00	-60.36	-62.65	-65.82	-63.82	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	$c_j - z_j$		60.36	62.65	65.82	63.82	0.00	-1.00	0.00	-1.00	0.00	-1.00	0.00	-1.00
Iteration 2														
1	a_1	0.26	0.01	-0.38	0.00	0.01	1.00	-1.00	0.00	0.00	0.00	0.00	-0.74	0.74
1	a_2	0.26	0.83	0.86	0.00	-0.07	0.00	0.00	1.00	-1.00	0.00	0.00	-0.74	0.74
1	a_3	0.13	-0.83	-1.53	0.00	-1.75	0.00	0.00	0.00	0.00	1.00	-1.00	-0.87	0.87
0	X_3	0.05	0.92	0.97	1.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.05	-0.05

	z_j	0.65	-0.01	1.06	0.00	1.80	1.00	1.00	1.00	1.00	1.00	1.00	4.35	-2.35
	$c_j - z_j$		0.01	-1.06	0.00	-1.80	0.00	-1.00	0.00	-1.00	0.00	-1.00	-3.35	2.35
Iteration 3														
1	a_1	0.15	0.72	0.93	0.00	1.51	1.00	-1.00	0.00	0.00	-0.85	0.85	0.00	0.00
1	a_2	0.15	1.53	2.16	0.00	1.42	0.00	0.00	1.00	-1.00	-0.85	0.85	0.00	0.00
0	δ_4	0.15	-0.96	-1.76	0.00	-2.01	0.00	0.00	0.00	0.00	1.15	-1.15	-1.00	1.00
0	X_3	0.06	0.87	0.88	1.00	0.89	0.00	0.00	0.00	0.00	0.06	-0.06	0.00	0.00
	z_j	0.30	-2.25	-3.09	0.00	-2.93	1.00	1.00	1.00	1.00	3.70	-1.70	2.00	0.00
	$c_j - z_j$		2.25	3.09	0.00	2.93	0.00	-1.00	0.00	-1.00	-2.70	1.70	-1.00	0.00
Iteration 4														
1	a_1	0.08	-0.19	0.00	-1.05	0.56	1.00	-1.00	0.00	0.00	-0.92	0.92	0.00	0.00
1	a_2	0.01	-0.60	0.00	-2.46	-0.78	0.00	0.00	1.00	-1.00	-0.99	0.99	0.00	0.00
0	δ_4	0.27	0.78	0.00	2.00	-0.22	0.00	0.00	0.00	0.00	1.27	-1.27	-1.00	1.00
0	X_2	0.07	0.99	1.00	1.14	1.02	0.00	0.00	0.00	0.00	0.07	-0.07	0.00	0.00
	z_j	0.09	0.80	0.00	3.51	0.21	1.00	1.00	1.00	1.00	3.91	-1.91	2.00	0.00
	$c_j - z_j$		-0.80	0.00	-3.51	-0.21	0.00	-1.00	0.00	-1.00	-2.91	1.91	-1.00	0.00

Iteration 5														
1	a_1	0.08	0.36	0.00	1.21	1.28	1.00	-1.00	-0.92	0.92	0.00	0.00	0.00	0.00
0	δ_3	0.01	-0.60	0.00	-2.47	-0.78	0.00	0.00	1.01	-1.01	-1.00	1.00	0.00	0.00
0	δ_4	0.27	0.02	0.00	-1.13	-1.21	0.00	0.00	1.27	-1.27	0.00	0.00	-1.00	1.00
0	X_2	0.07	0.95	1.00	0.97	0.97	0.00	0.00	0.07	-0.07	0.00	0.00	0.00	0.00
	z_j	0.08	-0.36	0.00	-1.21	-1.28	1.00	1.00	2.92	-0.92	2.00	0.00	2.00	0.00
	$c_j - z_j$		0.36	0.00	1.21	1.28	0.00	-1.00	-1.92	0.92	-1.00	0.00	-1.00	0.00
Iteration 6														
0	X_4	0.06	0.28	0.00	0.94	1.00	0.78	-0.78	-0.72	0.72	0.00	0.00	0.00	0.00
0	δ_3	0.05	-0.38	0.00	-1.73	0.00	0.61	-0.61	0.44	-0.44	-1.00	1.00	0.00	0.00
0	δ_4	0.35	0.36	0.00	0.01	0.00	0.94	-0.94	0.41	-0.41	0.00	0.00	-1.00	1.00
0	X_2	0.01	0.68	1.00	0.06	0.00	-0.76	0.76	0.76	-0.76	0.00	0.00	0.00	0.00
	z_j	0.00	0.00	0.00	0.00	0.00	2.00	0.00	2.00	0.00	2.00	0.00	2.00	0.00
	$c_j - z_j$		0.00	0.00	0.00	0.00	-1.00	0.00	-1.00	0.00	-1.00	0.00	-1.00	0.00

Phase 2 - Iteration 7														
1	X_4	0.06	0.28	0.00	0.94	1.00	0.78	-0.78	-0.72	0.72	0.00	0.00	0.00	0.00
0	δ_3	0.05	-0.38	0.00	-1.73	0.00	0.61	-0.61	0.44	-0.44	-1.00	1.00	0.00	0.00
0	δ_4	0.35	0.36	0.00	0.01	0.00	0.94	-0.94	0.41	-0.41	0.00	0.00	-1.00	1.00
1	X_2	0.01	0.68	1.00	0.06	0.00	-0.76	0.76	0.76	-0.76	0.00	0.00	0.00	0.00
	z_j	0.07	1.04	1.00	1.00	1.00	-0.03	0.03	-0.04	0.04	0.00	0.00	0.00	0.00
	$c_j - z_j$		-0.04	0.00	0.01	0.00	0.03	-0.03	0.04	-0.04	0.00	0.00	0.00	0.00
Iteration 8														
1	X_3	0.07	0.30	0.00	1.00	1.06	0.83	-0.83	-0.76	0.76	0.00	0.00	0.00	0.00
0	δ_3	0.17	0.13	0.00	0.00	1.84	2.04	-2.04	-0.88	0.88	-1.00	1.00	0.00	0.00
0	δ_4	0.35	0.35	0.00	0.00	-0.01	0.93	-0.93	0.42	-0.42	0.00	0.00	-1.00	1.00
1	X_2	0.00	0.66	1.00	0.00	-0.06	-0.81	0.81	0.81	-0.81	0.00	0.00	0.00	0.00
	z_j	0.07	1.04	1.00	1.00	1.01	-0.02	0.02	-0.05	0.05	0.00	0.00	0.00	0.00
	$c_j - z_j$		-0.04	0.00	0.00	-0.01	0.02	-0.02	0.05	-0.05	0.00	0.00	0.00	0.00

Table 4.15 Values Showing the Ranges of Investment Minimization Options

Variable	Value	Reduced Cost	Original Val	Lower Bound	Upper Bound
X_1	0	0.04	1	0.96	Infinity
X_2	0	0	1	0.94	1.03
X_3	0.07	0	1	0.97	1.01
X_4	0	0.01	1	1	Infinity
Constraint	**Dual Value**	**Slack/Surplus**	**Original Val**	**Lower Bound**	**Upper Bound**
Constraint 1	-0.02	0	1	0.92	1
Constraint 2	-0.05	0	1	1	1.09
Constraint 3	0	0.17	1	-Infinity	1.17
Constraint 4	0	0.35	1	-Infinity	1.35

This implies that for each value of X_1, X_2, X_3, and X_4 within the lower and upper bounds, the optimal solution will remain unchanged, regardless of the combinations of choices to minimize cost. The list of various feasible solutions is also shown in Table 4.16 below.

Table 4.16 Various Feasible Solutions of the Investment Minimization Options

Variable	Status	Value
X_1	NON-Basic	0
X_2	Basic	0
X_3	Basic	.07
X_4	NON-Basic	0

δ_1	NON-Basic	0
δ_2	NON-Basic	0
δ_3	Basic	.17
δ_4	Basic	.35
Optimal Value (Z)		**7.0**

Similarly, a series of combinations for the various options will still produce the same optimal solution of 7.0 within the boundaries.

CHAPTER 5

CONCLUSIONS AND RECOMMENDATION

5.0 Conclusions

This research is successful in modeling investment options of investors as a Game Theory problem that maximizes the returns from their investments. The application of Game Theory in the financial investment strategy is also successful in offering optimal solutions to an investor. The solution to these problems consisted of many feasible investment opportunities where an investor can invest, given that the total investment sum is not violated. Hence, the investor is able to make better investment policies based on which combinations of payoff is providing the optimal value of returns.

Furthermore, the results from the maximization of returns showed that, allocating 20 percent and 50 percent of the investor's funds in the first quarter and second quarter respectively, yielded the maximum returns to the investor. With an increase of 5% in Scenario 1, 4 percent and 3 percent more of investor's funds allocation in the first and second quarter respectively, yielded the maximum returns on the investment. With a similar increase of 5% in Scenario 2, the investor maximizes the most profit within July and September, whiles IGS Financial Services Limited minimizes the most cost in the 2^{nd} quarter of the year. The results from the minimization of cost however, showed that, allocating 70% of the investor's funds in a 12 months investment policy, minimizes the most cost to the company.

5.1 Recommendation

The use of Game Theory has proven to be efficient in the computation of optimum results and gives a systematic and transparent solution. It is therefore recommended that Invest Grow Secure Financial Services adopts and implements this strategy to help minimize their investment cost whiles maximizing the profits of an investor.

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