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A THESIS REPORT TITLED

MODIFICATIONS OF THE JANARDAN DISTRIBUTION AND ITS APPLICATIONS TO LIFETIME DATA

BY

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DECLARATION

I, Nelson Doe Dzivor, declare that this thesis is my own work. It is being submitted in fulfilment of the requirement for the award of the degree of Doctor of Philosophy in Mathematics (Statistics) at the University of Mines and Technology, UMaT, Tarkwa, Ghana. It has not been submitted for any degree or examination in any other university.

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ABSTRACT

One of the recently developed probability distributions which are gaining popularity in modelling is a two-parameter Janardan distribution. However, Janardan distribution is discovered to be limited in controlling skewness and kurtosis which most lifetime data exhibit hence the need to modify the Janardan distribution through the method of parametrisation. To improve the usability and flexibility of the Janardan probability distribution, the study is designed to come out with three new probability distributions of which Janardan distribution is a baseline, establish the statistical properties of the new distributions as well as test their goodness of fit through the use of data. In line with the study objectives, three new distributions are developed through the method of parametrisation. These new distributions are Exponentiated Janardan (Three parameter distribution), Kumaraswamy Janardan (Four Parameter distribution) and Exponentiated Kumaraswamy Janardan (Five parameter distribution). Statistical properties such as PDF, CDF, Hazard rate, Survivor rate, Moments, Moment Generating function and MLE are established for each of the derived distributions. Empirical results reveal that all the derived models provide a better fit to all the considered sample datasets than the existing sub-models. Apart from the fact that these three derived distributions show superiority over Janardan Distribution and its sub-model (Lindley distribution), the study further investigated the goodness of fit among the three new models. In comparing the three new distributions, the four-parameter Kumaraswamy Janardan (KJ) Distribution proves superiority in most cases. The researcher recommends that scholars should expand the statistical properties of the new distributions to bridge the research gap in mathematical computations. Also, industry experts in the field of reliability engineering, demography, actuary, etc; should use Kumaraswamy Janardan in modelling and predicting the reliability and hazard rate of their product since this distribution provides a robust hazard rate function.

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DEDICATION

I dedicate this thesis to God Almighty for His protection, wisdom, provision and guidance throughout my educational pursuits. I also dedicate this thesis to my wife (Dinah Owusu), my mother (Charity Ablavi Sakpetor) and my sons (Hamuel Makafui Dzivor, Lemuel Sitsope Dzivor and Kemuel Amenuveve Dzivor) for their sacrifices throughout my study period. Finally, I dedicate the thesis to my thesis supervisors for their valuable support from the start to the end of this thesis.

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TABLE OF CONTENTS

Content **Page**

Digitized by UMaT Library

Digitized by UMaT Library

LIST OF FIGURES

Digitized by UMaT Library

LIST OF TABLES

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CHAPTER 1

INTRODUCTION

1.1 Background of The Study

Though probability and statistics are applications of mathematics, probability is known to be the heart of statistics with a slight fundamental difference in worldview. While probability is concerned with the likelihood of future events, statistics analyses the occurrences of past events. The relevance, usefulness, and application of probability cannot be overestimated so far as life and life events are concerned. Modern probabilistic theories are developed from the dice table of France in 1654, "the gambler's dispute", (Camilleri, 2018).

Since 1654, many mathematicians (such as Pascal and Fermat, Christian Huygens, Jakob Bernoulli, Abraham de Moivre, Pierre de Laplace, Chebyshev, Kolmogorov, Poisson, Weibull, Kumaraswamy, Shanker, and others) have contributed immensely to the development of probabilistic theories. Probabilistic ideas have spread from the game of chance to many scientific and practical problems. Some of the important applications of probability theory are the theory of errors, actuarial mathematics, statistical mechanics, etc (Khan, 2016). Many more probability theory distributions are being developed from time to time. Even though the classical models serve great purposes in their area of application, they have not been able to stand the test of time (Khrennikov, 2019). This has motivated researchers to come out with modifications to these existing distributions to make them better. The Janardan is one such modification, but the distribution has limitations that make it difficult to apply it to real-life data.

The ability of distributions to effectively model data, especially lifetime data, is grossly influenced by parameters, thus; scale parameters and shape parameters. Scale parameters

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control the variability in the dataset while shape parameters control the kurtosis and skewness. However, some distributions do not have "sufficient" parameters to effectively model the lifetime dataset. An example of such distribution is the Janardan distribution.

Janardan distribution is a two-parameter distribution introduced by Shanker et al. (2013) of which the one-parameter Lindley (1958) distribution is a special case. Janardan probability distribution had been tried and tested on many lifetime datasets and proven to be a more relaxed and better fit than the Lindley distribution. However, since the introduction of Janardan, scholars (Elbatal *et al*., 2013; Warahena-Liyanage and Pararai, 2014; Shanker *et al*., 2014; Bashir and Rasul, 2016; Amer *et al*., 2017; Al-khazaleh *et al*., 2016; Hussian, 2014) have one way or the other tried to modify it and enhance its usability in diverse data environment through parametrisation. Diverse kinds of generators have been used by these scholars in adding parameter(s) to the Janardan distribution but none of them had used the Kumaraswamy generator despite its potential in improving "scaled-parametric" models.

The Kumaraswamy generator for parametrisation in this study was introduced by Cordeiro and Castro (2011). This generator has no scale parameter but has two shape parameters. This makes it suitable for modifying distributions that need shape parameters to be improved upon like the Janardan distribution.

1.2 Problem Statement

Janardan distribution has one scale parameter and one shape parameter. This means that it is limited in controlling skewness and kurtosis which most lifetime data exhibit. This distribution does not complement generators that lack shape parameters (Amer, 2017). In solving this problem, there is the need to modify the Janardan distribution by using a generator that has shape parameter(s). These modifications of Janardan distribution would be accomplished using Kumaraswamy and exponential distribution as generators. The

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Exponential distribution is used in introducing one additional shape parameter to the Janardan distribution. Kumaraswamy distribution is used to introduce two more shape parameters to increase reliability in controlling kurtosis and skewness. These modifications are to improve future usability and flexibility of Janardan probability distribution (which currently is rarely noticed in literature and application).

1.3 Research Questions

1. What are the new modifications of Janardan distribution that can fit lifetime data better than the Janardan model and its sub-models?
2. What are the parameter estimators of the new distributions of which the Janardan model is a particular case?
3. What are the effects of parameter estimators when simulated?
4. What are the modified distributions using real-time data?

1.4 Research of Objectives

The objectives of the study are:

1. To derive some new distribution from the Janardan distribution.
2. To derive parameter estimators of the new distributions.
3. To analyse the parameter estimators through simulation
4. To test the modified distributions using real-life data.

1.5 Organisation of the Thesis

This research work is organised into six (6) chapters. Chapter 1 is the introduction to the study while chapter 2 reviews various kinds of literature in connection to the study. Chapter 3 discusses the methodology used to achieve the objectives of the study. The theoretical

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results are presented in chapter 4 and the empirical result in chapter 5 while chapter 6 presents the discussion, conclusions, and recommendations of the study.

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CHAPTER 2

LITERATURE REVIEW

2.0 Introduction

According to Klakattawi (2019), Karl Pearson developed the Gamma family distributions in 1895. Uniquely among this family of distributions is the exponential distribution which came out clearly in 1930. This distribution is a continuous version of the Poisson distribution. Post-classical probability distributions are developed to better-fit lifetime data. However, these probability distributions are modifications that take their root from the classical probability distribution. This is because the classical models serve great purposes in their area of application. The exponential distribution (an example of a classical model) is the classical base distribution for this study. Many scholars have come out with modified versions of the exponential distribution, which are tested and established as a better-fit model than the exponential model. One of those probability models in which exponential distribution forms a sub-model is the Janardan distribution.

In this chapter, the study reviews sub-models of Janardan distributions, the Janardan distribution itself, and post-Janardan models. Also, Kumaraswamy's distribution and method of parameterisation are reviewed.

2.1 Pre-Janardan Distributions (Sub Janardan Models)

Pearson's gamma family of distributions published in 1895 had given birth to exponential distribution as unique distribution in 1930 (Klakattawi, 2019). Scholars such as Epstein (1958), Esary (1957), Ferguson (1964), Freund (1961), Gaver (1963), and Gumbel (1960), among others; carried out various studies on exponential distributions and found it to be a great distribution that was good in modeling data. Epstein (1958) discusses the role of exponential distribution in life testing in connection with industrial quality control. In 1957, Esary discussed the appropriateness of exponential distribution in modeling the stochastic theory of accident and survival. Ferguson (1964) vividly explains the mathematical characteristics of the exponential distribution. In improving Gumbel's work, Freund (1961) came out with a bivariate extension of the exponential distribution. The exponential distribution is very central in modeling in various fields ranging from engineering, quality control, medicine, bio-chemical, and computer science among others. Weibull (1939)

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suggests the relevance of exponential distribution in modeling actuarial, biological, and engineering problems.

Despite the relevance of exponential distribution as presented by many scholars, Marshall (1966) questioned the assumptions of the exponential distribution, especially, in life situations where assumptions are found to be questionable or false. In addressing this novelty, Marshall came out with meaningful derivations of the multivariate exponential distribution. This is a build-up of the work of Gumbel (1960) and Freund (1961). So, it is very clear that the need to improve on exponential distribution dated decades ago.

Recently, the dynamism of the knowledge wave around exponential distribution has changed. Many researchers have extended exponential distribution by using the distribution as a generator for parametrisation.

As an extension of the exponential distribution, Weibull (1939) proposed a two-parameter Weibull distribution with cumulative density function as:

$$X \sim Weibulldistribution$$

$$F(x) = 1 - e^{\left(\frac{x}{\lambda}\right)^{k}}; x, k, \lambda > 0 \qquad (2.1)$$

This distribution approximates to exponential distribution if the shape parameter (k) is set to 1. Hence the CDF for exponential distribution is given as

$$F(x) = 1 - e^{\left(\frac{x}{\lambda}\right)^{1}}; x, \lambda > 0 \qquad (2.2)$$

Deducing from equations (2.1) and (2.2), clearly demonstrates that exponential distribution is a special case of Weibull distribution. Hence, the exponential distribution is considered a sub-model of the Weibull model. It would not be wrong to conclude that the Weibull distribution is exponentiated exponential distribution. This distribution has many applications in reliability engineering and survival analysis as well as industrial quality control (Nikulin, 2013).

Very important functions in stochastic modeling in the area of survival and reliability analysis are survivor function and hazard function. Hazard function models the risk of failure in an infinitesimally small time between $t + \partial t$. The theoretical hazard function of the Weibull distribution is given in (2.3).

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$$h(t) = \frac{k}{\lambda^k} t^{k-1} \quad (2.3)$$

Setting k = 1, the hazard function of Weibull approximates the hazard function of exponential distribution (as demonstrated in 2.4).

$$h(t) = \frac{1}{\lambda} \quad (2.4)$$

From equation 2.4, it is demonstrated that h(t) solely depends on the model scale parameter and is constant over time. The assumption of constant hazard over time is often not realistic. In solving this drawback in exponential distribution, Weibull introduced the shape parameter. This transforms the hazard function into a monotonic hazard function, which can either be decreasing or increasing but not both. Again, the assumption of monotonicity may not be feasible in some cases.

Another major extended exponential distribution is the Lindley distribution. Lindley distribution is introduced by D.V Lindley in 1958. Lindley's (1958) distribution is a build-up of the exponential distribution. According to Sah (2019), “exponential and Lindley occupy central places among the class of continue probability distributions and play important roles in statistical theory”. Mishra and his team have spent quite a significant amount of time and effort on the modification of Lindley distribution. Mishra and Sah (2016) introduced Generalised Exponential-Lindley distribution (GELD). This model is found to provide a better fit to most life data than the Exponential-Lindley model. In an attempt to discretise a continuous distribution, Mishra and Sah (2019) introduced a distribution that merges GELD with Poisson and termed it as Generalised Exponential-Lindley Mixture of Poisson Distribution (GELMPD). This new modification was tested on lifetime data and compared with Sankaran's (1970) Poisson Lindley Distribution (PLD) and has proven that it gives a better fit than PLD. A lot of scholars and researchers have contributed to improving Lindley distribution. Ghitany et al. (2008, 2009, 2011) and Bakouch et al. (2012) are among numerous researchers who have discussed the properties, relevance, and application of Lindley distribution. Shanker (2013), Zakerzadeh and Dolati (2009) introduced higher parameter distributions and their various applications and properties were discussed.

There have been a lot of modifications on this Lindley Distribution. Among these modified distributions is Shanker’s Janardan distribution of which Lindley is the baseline distribution.

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Janardan Distribution is a two-parameter version of Lindley distribution with Cumulative Density Function as:

$$F(x;\alpha,\theta) = 1 - \left[\frac{1+\alpha\theta+\theta x}{\alpha\theta+1}\right]e^{-\theta x} \quad (2.5)$$

Differentiating the CDF gives the probability density function as in equation (2.6)

$$f(x;\alpha,\theta) = \frac{\theta^2}{\theta+\alpha}(1+\alpha x)e^{-\theta x} \quad (2.6)$$

Setting $\alpha = 1$, one parameter Lindley distribution is obtained. The CDF and PDF of one parameter Lindley distribution are given in equations (2.7) and (2.8).

$$F(x;\theta) = 1 - \left[\frac{1+\theta+\theta x}{\theta+1}\right]e^{-\theta x}, x, \theta > 0 \quad (2.7)$$

$$f(x;\theta) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x} \quad (2.8)$$

Setting $\alpha = 0$, The Lindley distribution approximates the exponential distribution.

$$f(x;0,\theta) = \theta e^{-\theta x} \quad (2.9)$$

Hence, it is seen that exponential distribution in equation (2.9) is a baseline model of Lindley Distribution in equation (2.8). Also, Lindley Distribution in equation (2.8) is a baseline distribution of Janardan Distribution in equation (2.6).

The expansion and modification of Lindley distribution continue to occupy space in the literature and knowledge gap. Gómez-Deniz (2011) discuss the properties and applications of discrete Lindley distribution.

2.2 Janardan Distribution

Ghitany et al. (2008) discussed some significant statistical properties of one one-parameter Lindley (1958) distribution. The study contributes significantly to the success of Shanker et al (2013) two-parameter distribution known as the Janardan Distribution. Shanker and his

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co-researchers introduced Janardan Distribution in 2013. They established the Cumulative density function and probability density function. They further went on to establish some statistical properties of the new distribution. Some of the statistical properties established by Shanker et al. (2013) and the team are the mode of the distribution, the moments and their related measures, the failure rate, and the mean residual life, stochastic orderings as well as estimation of parameters. They concluded their work by applying the distribution to several life data and found out that the Janardan distribution provides closer fits than the Lindley distribution.

For a random variable that follows Janardan Distribution (JD) with parameters α, θ; the probability density function is given as:

$$X \sim JD(x, \theta, \alpha) = f(x, \theta, \alpha).f(x, \theta, \alpha) = \frac{\theta^2}{\alpha(\theta + \alpha^2)}(1 + \alpha x)e^{-\frac{\theta}{\alpha}x};$$

$$x, \alpha, \alpha, \theta > 0 \qquad (2.10)$$

Integrating the pdf, Shanker established the cumulative density function of Janardan as

$$F(x) = 1 - \frac{\alpha(\theta+\alpha^2)+\theta\alpha^2 x}{\alpha(\theta+\alpha^2)} e^{-\frac{\theta}{\alpha}x} \quad ; x > 0, \alpha > 0, \theta > 0 \qquad (2.11)$$

The mean or expectation of Janardan distribution (which is also known as the first moment) is also established as:

$$E(X) = \mu_X = \frac{\alpha(\theta+2\alpha^2)}{\theta(\theta+\alpha^2)} \qquad (2.12a)$$

Also, the second, third, and fourth moments are appropriately established as follows:

$$\mu_2 = 2\left[\frac{\alpha}{\theta}\right]^2 \frac{(\theta + 3\alpha^2)}{(\theta + \alpha^2)}$$

$$\mu_3 = 6\left[\frac{\alpha}{\theta}\right]^3 \frac{(\theta + 4\alpha^2)}{(\theta + \alpha^2)}$$

$$\mu_4 = 24\left[\frac{\alpha}{\theta}\right]^4 \frac{(\theta + 5\alpha^2)}{(\theta + \alpha^2)}$$

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By generalisation, the r[th] moment is given as

$$\mu_r = r!\left[\frac{\alpha}{\theta}\right]^r\left[1+\frac{r\alpha^2}{\theta+\alpha^2}\right]; r = 1,2,3,\ldots \tag{2.12b}$$

The mode of the distribution is established as:

Differentiating the pdf and setting the result to zero, the extrema of the Janardan distribution.

$x = \frac{\alpha^2-\theta}{\alpha\theta}$. This value, the $f(x)$ is maximum only if $\alpha > \theta$. But in a situation where $\alpha < \theta$

The $f(x)$ is decreasing in X. Therefore, the mode of Janardan distribution is established to be:

$$\begin{cases} \frac{\alpha^2-\theta}{\alpha\theta}, \alpha>\theta \\ 0, otherwise \end{cases}$$

Furthermore, Shanker and the team established the Hazard function $h(x)$ and mean residual rate $m(x)$ respectively as follows:

$$h(x) = \frac{f(x)}{1-F(x)} = \frac{\frac{\theta^2}{\alpha(\theta+\alpha^2)}(1+\alpha x)e^{-\frac{\theta}{\alpha}x}}{\frac{\alpha(\theta+\alpha^2)+\theta\alpha^2 x}{\alpha(\theta+\alpha^2)}e^{-\frac{\theta}{\alpha}x}}$$

$$= \frac{\frac{\theta^2}{\alpha(\theta+\alpha^2)}(1+\alpha x)}{\frac{\alpha(\theta+\alpha^2)+\theta\alpha^2 x}{\alpha(\theta+\alpha^2)}} = \frac{\frac{\theta^2}{1}(1+\alpha x)}{\frac{\alpha(\theta+\alpha^2)+\theta\alpha^2 x}{1}} =$$

$$= \frac{\theta^2(1+\alpha x)}{\alpha(\theta+\alpha^2)+\theta\alpha^2 x}$$

$$h(x) = \frac{\theta^2(1+\alpha x)}{\alpha(\theta+\alpha^2)+\theta\alpha^2 x} \tag{2.13}$$

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$$m(x) = \frac{1}{1 - F(x)} \int_x^\infty [1 - F(t)] dt$$

Therefore, $m(x) = \frac{\alpha}{\theta}\left[\frac{(\theta+\alpha^2)+(\theta\alpha x+\alpha^2)}{(\theta+\alpha^2)+\theta\alpha x}\right]$ (2.14)

Janardan probability distribution had been tried and tested on many lifetime datasets and proven to be a more relaxed and better fit than the Lindley distribution.

However, since the introduction of Janardan (a generalisation of Lindley distribution), a series of studies are carried out in the bid to propose a distribution that would better fit lifetime data and be more flexible.

Lindley distribution has attracted many researchers and industrial players in modeling organisational and industrial lifetime data. Many generalisations of the Lindley distribution are done and hence providing competing distributions with the Janardan distribution. Below are some generalisations of Lindley distribution (that are competing with Janardan).

Sankaran (1970) introduced the Poisson-Lindley distribution for modeling discrete random variables. This distribution has a probability mass function as:

$$P(X = x) = \frac{\theta^2(\theta+x+2)}{(\theta+1)^{x+3}}; x = 0,1,\ldots; \theta > 0 \tag{2.15}$$

Another competing distribution with Janardan (as a Lindley generalised) is the Generalised Lindley Distribution (GLD). This distribution is introduced by Zakerzadeh and Dolati (2009). This distribution has three non-negative parameters; $\theta$and γ (scale parameters) and α (shape parameter) and has a probability density function as:

$$f(x) = \frac{\theta^2(\theta x)^{\alpha-1}(\alpha+x)e^{-\theta x}}{(\gamma+\theta)\Gamma(\alpha+1)}; x > o, \alpha > 0, \theta > 0; \gamma > 0 \tag{2.16}$$

Another modification to the Lindley distribution is “The Negative Binomial-Lindley Distribution. This distribution was introduced by Zamani and Ismail (2010). This is a discrete model developed by merging negative binomial distribution and Lindley distribution. The new model has a probability mass function (pmf) as:

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$$p(x) = \frac{\theta^2}{\theta+1}\binom{r+x-1}{x}\sum_{j=0}^{k}\binom{k}{j}(-1)^{j}\frac{\theta+r+j+1}{(\theta+r+j)^2}; x = 0,1,2,\ldots; \theta > 0 \quad (2.17)$$

Another generalisation of Lindley distribution is Extended Lindley Distribution (EXLD) introduced by Bakouch et al. (2012). This distribution has CDF as:

$$F(x) = 1 - \left[\frac{1+\theta+\theta x}{1+\theta}\right]^{\alpha} e^{-(\theta x)^{\beta}}; x > 0, \theta > 0, \alpha \in \mathbb{R}^{-1} \quad (2.18)$$

Shanker and Mishra (2013) introduced "the Quasi Lindley Distribution" (QLD) as a generalisation of Lindley distribution with pdf as:

$$f(x) = \frac{\theta(\alpha+\theta x)}{\alpha+1} e^{-\theta x}; x > 0, \theta > 0, \alpha > 0 \quad (2.19)$$

Elbatal et al. (2013) also introduced New Generalised Lindley Distribution (NGLD) with the proposed PDF as:

$$f(x) = \frac{1}{1+\theta}\left[\frac{\theta^{\alpha+1}x^{\alpha-1}}{\Gamma(\alpha)} + \frac{\theta^{\beta}x^{\beta-1}}{\Gamma(\beta)}\right] e^{-\theta x}; x > o, \alpha > 0, \theta > 0; \beta > 0 \quad (2.20)$$

The last but not the least competing model considered in this review is "The Generalised Power Lindley Distribution" (GPLD). This distribution was introduced by Warahena-Liyanage and Pararai (2014). It is an improvement on Ghitany et al. (2005)'s Power Lindley Distribution (PLD). The CDF was gotten through the method of exponentiation.

$$F(x) = \left\{1 - \left[1 + \frac{\beta x^{\alpha}}{\beta+1}\right] e^{-\beta x^{\alpha}}\right\}^{\omega}, x, \beta, \alpha, \omega, > 0 \quad (2.21)$$

All these distributions are diligently derived and tested on datasets. However, some of them fit the data better than others. Despite their ability to fit data, they all have their weaknesses.

2.3 Post-Janardan Reviews

Since the introduction of Janardan distribution by Shanker et al. (2013), Shanker and his team have been working to improve the relevance as well as usability of the distribution.

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Shanker et al. (2014) introduced a more general form of the Janardan distribution by merging Poisson and Janardan to arrive at the Discrete Poisson Janardan Distribution (PJD) in which Sankaran's (1970) discrete Poisson Lindley Distribution (PLD) was a special case. Statistical properties such as the first four moments of PJD are established and parameters of the distribution are estimated using the moment method as well as the maximum likelihood estimation approach. The PJD was tested on some datasets of which PLD was used to model. With regard to the goodness of fit, PJD was found to provide a closer fit than PLD.

Bashir and Rasul (2016) did a lot of work on some properties of the Size-Biased Janardan Distribution, of which the Size-Biased Lindley Distribution is a special case. In their publication, they conclude that "Janardan Distribution is one of the important distributions for lifetime model and it has many applications in real life data". They established statistical properties such as moments, median, skewness, kurtosis, and Fisher index of dispersion of a Size-Biased Janardan Distribution. This distribution was tested on life data and compared Size-Biased Lindley distribution and was found to be better and more flexible in reliability analysis. The hazard rate was derived and found to be monotonically increasing.

Amer et al. (2017) introduced a generalisation of Janardan distribution known as Transmuted Janardan distribution (TJD). These researchers made use of Shaw and Buckley's (2009) quadratic rank transmutation map to generate TJD. Parametrisation through transmutation is not limited to Amer and his co-researchers. Scholars (Elbatal & Elgarhy, 2013; Aryal & Trokos, 2009; Al-khazaleh *et al*., 2016; Elbatal *et al*, 2014; Hussian, 2014; etc) have improved one distribution or another through transmutation.

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Table 2.1 Exponential Generalised Distributions – Strengths and weaknesses

The distributions	Parameter	Description	Problem
Exponential Distribution	θ	This distribution has one scale parameter with no shape parameter.	This distribution cannot control skewness and kurtosis which most lifetime data exhibit. Also, it has a constant hazard rate (which is unrealistic)
Weibull Distribution	λ, k	It is an improvement on exponential distribution. It has one scale parameter (λ) and one shape parameter k.	This distribution can model variability, skewness, and kurtosis which most lifetime data exhibit. Also, it has a monotonic hazard rate (which is unrealistic with lifetime data)
Lindley Distribution	θ	It is an improvement on exponential distribution. It has one scale parameter (θ).	This distribution has a good hazard rate. However, it has no shape parameter to be able to control kurtosis and skewness.
Janardan Distribution (a two-	θ, α	It is an improvement on Lindley distribution. It has	This distribution has a good hazard rate and lifetime data better than Lindley. However, it has

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parameter Lindley distribution		scale and shape parameters (θ, α).	limitations in controlling kurtosis and skewness.
Generalised Lindley Distribution (GLD)	θ,γ, and α	This is an improvement on Janardan distribution. It has three parameters; $\theta$and γ (scale parameters) and α (shape parameter).	Even though this distribution has one shape parameter to control kurtosis and skewness to some extent, there is still the need to improve on it to make it more accommodating.
Extended Lindley Distribution (EXLD)	θ, α	This is a competing model with the Janardan model with two scale parameters (θ, α).	This distribution has a good hazard rate and lifetime data better than Lindley. However, it has no shape parameter to be able to control kurtosis and skewness.
New Generalised Lindley Distribution (NGLD)	α, θ, β	This distribution has one scale parameter and two shape parameters. This means that the distribution can control kurtosis and skewness to some extent as well as variability.	This distribution can further be modified to improve on its ability to model lifetime data better.

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The Generalised Power Lindley Distribution (GPLD)	$\beta, \alpha, \omega, > 0$	This distribution has one scale parameter and two shape parameters. This means that the distribution can control kurtosis and skewness to some extent as well as variability.	This distribution can further be modified to improve its ability to model lifetime data better.

2.4 Methods of Developing New Probability Distributions

The quest for the precision of prediction has increased in recent years. Due to that, the medium of prediction has evolved. Probability distribution, been a medium for prediction, has witnessed a wave of new distributions of which existing ones are baselines. However, Pinho (2017) proposed two methods. He proposed that new probability distributions could be built using the computer-based method as well as the composite method. Hao (2014), in his doctoral dissertation, came out with new probability distributions based on random extrema and permutation patterns. Khalil et al. (2021) suggested and used the method of parametrisation and described it as a novel method for developing efficient probability distribution. In this section of the study, the computer-based method, composition/parametrisation method, random extrema method, and permutation patterns method are discussed.

2.4.1 Method of New Distribution – Computer-based Method

One of the methods of building new probability distribution as discussed by Pinho (2017) is a computer-based method. This approach of probabilistic modeling is done using computer-based evolutionary algorithms to come out with a customised cumulative density function (CDF) based on a given data. It is a kind of experimental approach. According to Pinho (2017), this approach of developing a new distribution works when there is limited understanding of the given problem to permit the building of an explicit CDF or/and to

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propose the use of an existing CDF. This approach attempts to find the "best possible" continuous probability distribution to model the given data concerning some fitness and optimality criteria and thereby controlling the mathematical complexity of outcome cumulative density function (CDF). The main baseline algorithm of this method is the Probabilistic Incremental Program Evolution (PIPE) algorithm which was developed by Salustowicz and Schmidhuber (1997).

Salustowicz and Schmidhuber (1997) presented the PIPE algorithm which is capable of producing programs according to to set rules governing probability. These probability rules are bettered over several iterations for the resultant programs are more probably to solve a given problem. This set of rules in a certain order is dependent on nodes and edges with certain assumptions. For instance, the PIPE, in its original proposition, assumes normality and exponentiality of continuous variables. The uniqueness of the PIPE algorithm is continuous parsing the program to a tree known as Probabilistic Prototype (PPT) and then randomly traversing on the PPT to sets of programs.

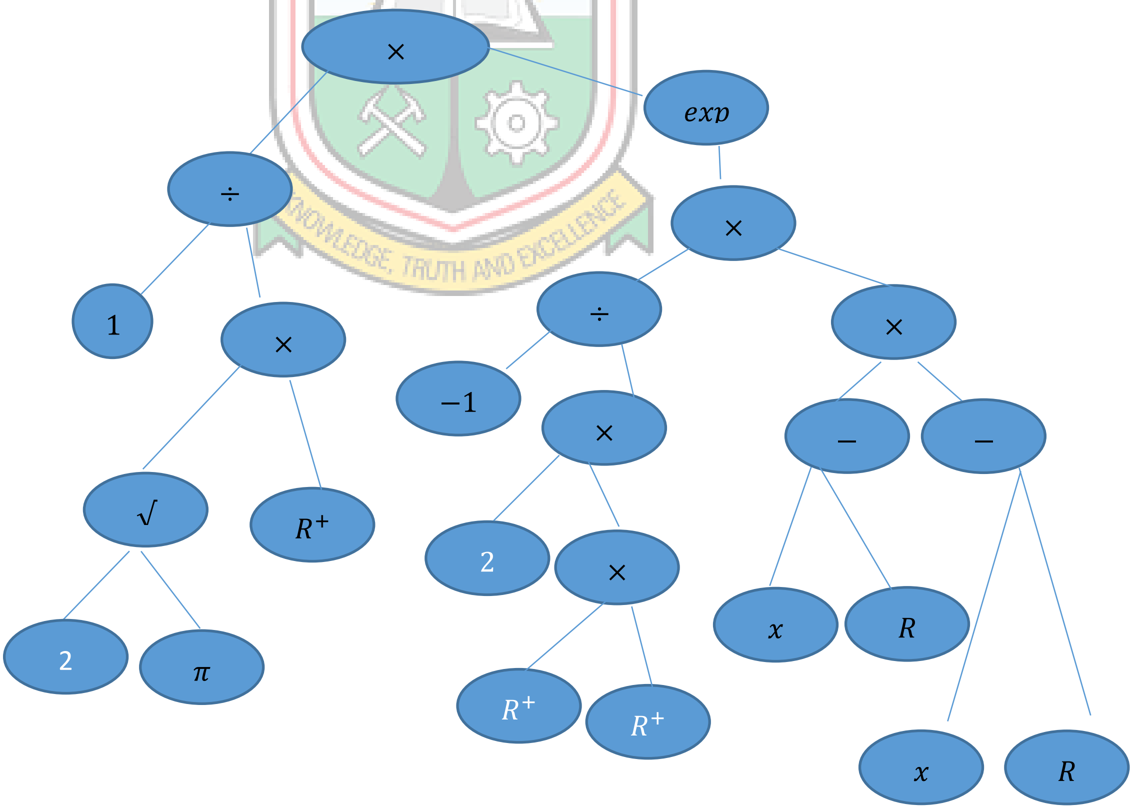

Figure 2.1 An example of a Probabilistic Prototype assuming normality

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From figure 2.1, PPT presents the "n" maximum arity of the instruction in the given function, F and terminal, T. The PPT contains "nodes" ($N_{d,w}$) at "depth" ($d \geq 0$) and "widths" ($w \geq 0$) with a probability vector ($P_{d,w}$). The entries of $P_{d,w}$ are given as $P_{d,w}(I)$ for every $I \in F \cup T$. The eligibility check of the nodes should satisfy the probability condition as given as:

$$\sum_{l,\ l \in F \cup T,}^{n} P_{d,w}(I) = 1;\ \ \forall N_{d,w} \qquad (2.22)$$

In a case when PPT is traversed in a depth starting at a node, an instruction "l" is auto-selected at the accessed node with a probability, with certain threshold conditionality, thereby creating a new "Tree". Recursively, new nodes are added (growing), some old ones are retained, and some are deleted (pruning) at each iteration.

Critical analysis of PIPE led to the discovery of two learning mechanisms namely: elitist learning and generation-based learning. These two learning mechanisms alternate until an ending criterion is met. The major part of the learning mechanism, generation-based learning, is known to have five distinguishable phases. Phase one is the creation of a program population while Phase two deals with population evaluation based on a fitness function. The next phase is learning from the population based on modified likelihood at each node while phase four deals with mutation of the PPT and the final phase is for PPT pruning.

Besides Pinho (2017), other scholars (Pelikan *et al*., 2002; Ondas *et al*., 2005) have applied and presented PIPE in their scholastic publications.

2.4.2 Method of New Distribution – Random Extrema Method

Hao (2014) employed the method of extrema and permutation patterns to come out with several new probability distributions. In his work, he studied a new family of distributions of random variables which are double-bounded [0, 1], and satisfied the condition of iid (identically and independently distributed). The two new distributions proposed by this author are the "Standard Uniform Geometric Model" and the "Correlated Standard Uniform Geometric Model". The models are a build-up of Uniform Distribution.

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The standard Uniform Geometric model's pdf is established as follows:

$$g(y) = \frac{\theta}{[1-(1-\theta)y]^2}; \quad 0 < y < 1$$

Its expectation is as follows:

$$E(Y^k) = \frac{\theta}{[1-\theta]^{k+1}} \sum_{j=0} \binom{k}{j} \int_0^1 (-u)^{j-2} du \;\; ; \;\; j = 0,1,\dots,k \; and \; u = 1-(1-\theta)y$$

2.4.3 Parameterisation of Distribution – Composition Method of New Distribution

Parameterisation is a method of adding parameter(s) to a distribution. This is done by merging two or more probability distributions through the generator. By so doing, a new distribution is formed from the existing distribution. This is the recent trait of distribution modifications due to the rate at which data is generated. The quantum of data available today is growing increasingly faster.

Mudholkar et al., (1993) introduced a one-parameter generator known as an exponential generalised generator with cdf given as: $F(x) = [G(x)]^{\alpha}$, $\alpha > 0$. This generator helps in improving a distribution that has only scale parameter by introducing a shape parameter $\alpha > 0$

Eugene et al. (2002) defined beta generalised generator. This generator improves on existing distribution by introducing additional three parameters. The CDF of this generator is given as:

$$F(x) = \frac{1}{B(a,b)} \int_0^{G(x)} t^{a-1}(1-t)^{b-1}\, dt \text{ , where } a > 0; b > 0 \text{ and } 0 < t < 1 \qquad (2.22)$$

Cordeiro and Castro (2011) introduced Kumaraswamy G1 generator with CDF as

$$F(x) = 1-(1-G^a(x))^b; a, b > 0 \qquad (2.23)$$

Ramos (2019) also introduced three-parameter Kumaraswamy generator known as Kumaraswamy-G exponentiated generator with CDF as

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$$F(x) = (1 - (1 - G^a(x))^b)^c; a, b, c > 0 \quad (2.24)$$

Other generators such as Gamma-G type 1 (Zografos & Balakrishnan, 2009), Odd-gamma-G type 3 (Torabi & Montazari, 2012), Logistic-G (Torabi & Montazari, 2014), Transformed-Transformer, T-X (Alzaatreh et al, 2013), Exponentiated T-X (Alzaghal *et al.*, 2013), Odd exponentiated generalised (Cordeiro et al, 2013), Exponentiated half-logistic (Cordeiro *et al.*, 2015), Kumaraswamy-G (Das, 2012); are among generators which are commonly used in distribution improvement lately.

2.5 Kumaraswamy Distribution

In 1980, Kumaraswamy introduced Kumaraswamy distribution as a better substitute for beta distribution. This was introduced when beta distribution failed to model and explain hydrological applications (Khan *et al.*, 2020). Kumaraswamy compared his distribution to some widely used distributions at the time and ran them on hydrological data and found out that his distribution better fit hydrological data. Among the competing distributions are log-normal distribution, beta distribution, and normal distribution (Kumaraswamy, 1980). This two-parameter distribution (Kumaraswamy) has been applied, by many scholars, in modeling test scores, height data, the temperature of the atmosphere, and many more (Jones, 2009). Since its introduction, many scholars (Khan *et al.*, 2020; El-Sherpieny, 2014; among others) have presented Kumaraswamy (in their research works) to be better distribution than its computing distributions. Hence, Kumaraswamy is the transformed transformer generator for most distributions that need modification to enable them to model closely current lifetime data in this data age. That is the motivation for the researcher to choose Kumaraswamy as a generator to improve Janardan distribution.

Kumaraswamy generalised distribution is one of the distributions that use the Kumaraswamy generator to come out with a five-parameter distribution. This new family of distribution was introduced by Das D. (2012). This new family of Kumaraswamy distribution has some sub-models under it. One of the sub-models is Kumaraswamy Normal distribution (Kw-N). This sub-model has one location parameter, one scale parameter, and two shape parameters. The PDF of Kumaraswamy's normal distribution is given by Das (2012) as:

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$$f(x) = \frac{ab}{\sigma}\phi(\frac{x-\mu}{\sigma})\{\Phi(\frac{x-\mu}{\sigma})\}^{a-1}\{1-\Phi(\frac{x-\mu}{\sigma})^{a}\}^{b-1} \quad (2.25)$$

$$a, b, \mu, \sigma^2$$

When the location parameter, μ, is set to zero and the scale parameter, σ, is set to unitary, the distribution transforms into a Standard Kumaraswamy-Normal distribution. Again, when a =2 and b = 1, Kumaraswamy's Normal distribution is reduced to a skew-normal distribution with the shape parameter being unitary.

Another special case of the Kumaraswamy-Generalised family of distributions introduced by Das (2012) is the Kumaraswamy-Weibull distribution. This is a four-parameter distribution with PDF as

$$f(x) = abc\beta^{c}x^{c-1}\, exp\{-(\beta x)^{c}\}[1-exp\{-(\beta x)^{c}\}]^{a-1} \times \{1-[1-exp\{-(\beta x)^{c}\}]^{a}\}^{b-1}. \quad (2.26)$$

It is very conspicuous that when "c" is set to one, the whole distribution is reduced to Kumaraswamy-Exponential distribution.

Yet another special case of the Kumaraswamy-Generalised family of distributions introduced by Das (2012) is the Kumaraswamy-Gamma distribution. This is a four-parameter distribution with PDF as

$$f(x) = \frac{ab\beta^{\alpha}x^{\alpha-1}e^{-\beta x}}{\Gamma(\alpha)^{ab}}\Gamma_{\beta x}(\alpha)^{a-1}\{\Gamma(\alpha)^{a}-\Gamma_{\beta x}(\alpha)\}^{b-1} \quad (2.27)$$

Kumaraswamy is a widely used generator in developing probability distributions. Abd Al-Fattah et al. (2021) used Kumaraswamy as a generator to introduce Exponentiated Generalised inverted Kumaraswamy distribution and established some statistical properties and applications. Mostafa et al. (2014) estimate parameters of Kumaraswamy distribution using general progressive type II censoring. El-Sherpieny et al. (2014) introduced Kumaraswamy-Kumaraswamy. Also, generalised order statistics from Kumaraswamy distribution were established by Garg (2009). Chakraborty et. al (2020) published a scholastic article on the Kumaraswamy Poisson-G family of distribution. Khan et al. (2016) proposed transmuted Kumaraswamy distribution using Shaw et al (2009) quadratic rank transmutation map.

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2.6 Summary of the Review

Many scholars have demonstrated in their publications the special and significant role exponential distribution plays in modeling in the fields of actuary, biology, engineering, and many others.

Some scholars (though, admitting the relevance of exponential distribution) also questioned some of its assumptions. This questioning of assumptions of exponential distribution dated way back to 1966. This spark of dynamic knowledge waves around exponential distribution. Many scholars came out with improved versions of Exponential distribution. Among the modified versions of exponential distribution are The Weibull distribution, Lindley distribution, and Lindley generalised distribution (of which Janardan distribution is a particular case).

In efforts to enhance the goodness of fit of Exponential/Lindley distribution, Shanker (2013) established (through theoretical and empirical analysis) that Janardan distribution fits lifetime data better than pre-Janardan distributions reviewed. Many scholars (through the method of parameterisation) established competing models. These competing models together with the Janardan model were all found out having a monotonically increasing hazard rate hence proving the basis for further improvement.

The review also critically examined several generators which aided scholars to come out with modified distributions through the method of parameterisation. Among the generators reviewed is the Kumaraswamy generator which is of significant essence to this study. Many scholars had used the Kumaraswamy generator to improve on existing distributions and achieved better results.

Therefore, the weaknesses in the Janardan model and its competing models coupled with potential in the Kumaraswamy generator provide the basis for coming out with better distribution in this study.

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CHAPTER 3

METHODS USED

3.0 Introduction

This section presents the methods used in this study. New distributions are developed using parameterisation. The parameters are estimated using maximum likelihood estimation with the help of simulated annealing. This section addresses the methods under the following sub-sections.

3.1 Method of parameterisation

Method of parameterisation (for sometimes now) is a new technique useful in developing new probability distribution by many statisticians and mathematical scientists.

To correct the limitations of Janardan distribution in handling lifetime data, three modified distributions are developed and tested with lifetime data. The proposed distributions are Exponentiated Janardan distribution (EJ), Kumaraswamy Janardan distribution (KJ), and Exponentiated Kumaraswamy Janardan Distribution (EKJ). These new modified models are developed through the method of parameterisation.

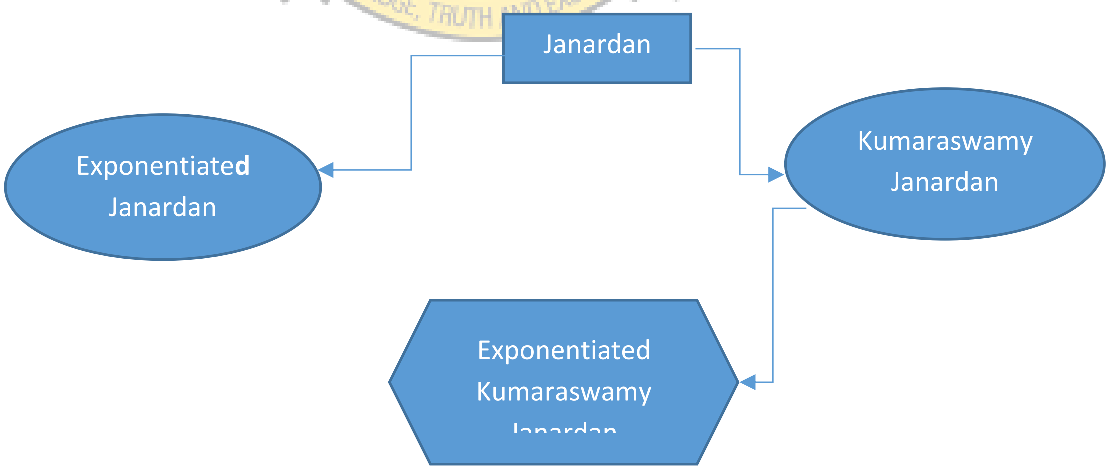

Figure 3.1 Flowchart Demonstrating Parameterisation of The New Models

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From the above flowchart, it is clear that the baseline distribution for the proposed distributions is a generalised Lindley distribution known as the Janardan distribution. The parameterisation for the proposed model 1 is diagrammatically presented below.

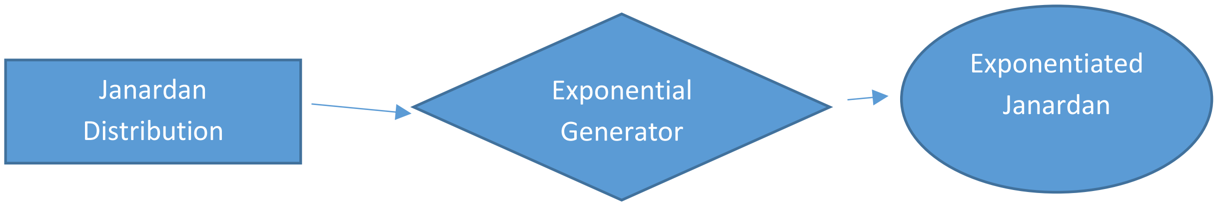

Figure 3.2 Flowchart Demonstrating Parameterisation Leading to Exponentiated Janardan

The parameterisation for proposed model 2 is diagrammatically presented below.

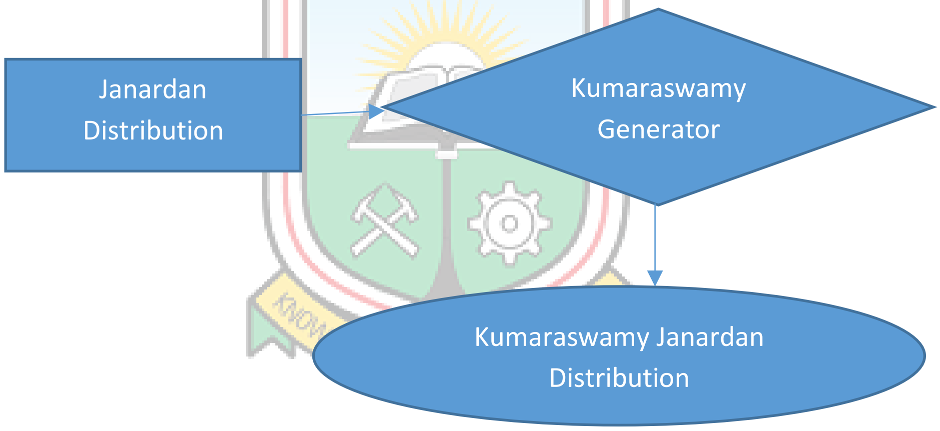

Figure 3.3 Flowchart Demonstrating Parameterisation Leading to Kumaraswamy Janardan Model

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The parameterisation for proposed model 3 is diagrammatically presented below.

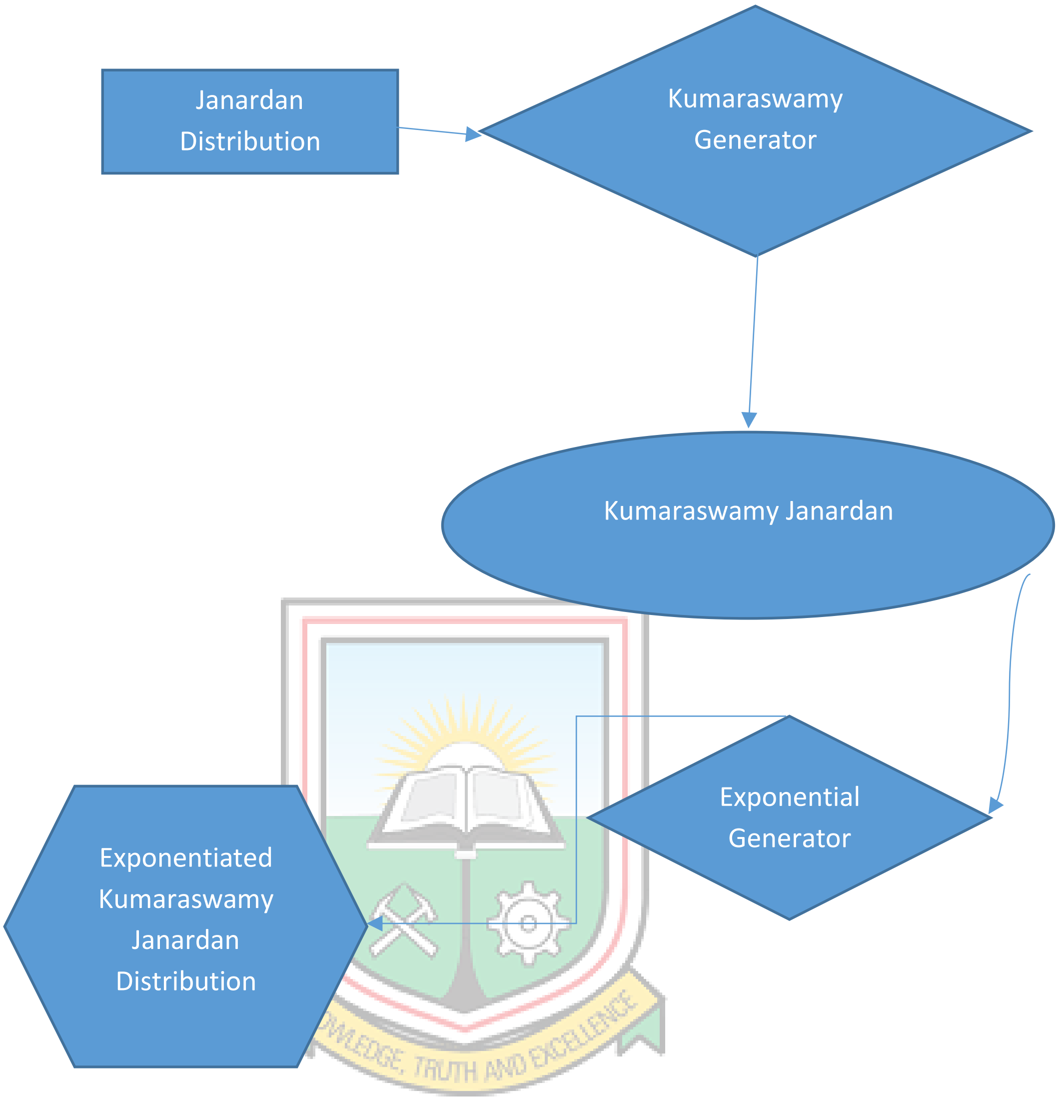

Figure 3.4 Flowchart Demonstrating Parameterisation Leading to Exponentiated Kumaraswamy Janardan Model

For each of the proposed models, Cumulative Density Function (CDF), Probability Density Function (PDF), hazard rate function, reliability function, moments and moment generating function as well as maximum likelihood function are derived. The proposed new modified distributions together with the sub-models are tested on lifetime data. The goodness of fit of the distributions is tested using Akaike Information Criterion (AIC).

3.2 Optimisation Techniques

This section presents the theoretical background of the optimality technique used in the study. The optimality technique employed in the study is maximum likelihood estimation.

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In probability, Maximum Likelihood Estimation is used to estimate parameters of probability distribution given some observed sample data. According to Rossi (2018), in maximising the likelihood function of the probability model, the sample data is most probable. The estimated point in the parameter space that maximises the likelihood function is called the maximum likelihood estimate. Any given set of observations is a sample from an unknown population, and MLE is to help make inferences about the population that is most likely to have generated the sample (Myung, 2003).

Consider a random sample $X = (X_1, X_2, \dots, X_n)$ being independent and identically random variables, each with probability density function as $f(xI\theta_X)$

Where $f(xI\theta_X) \in Ƒ$

and $Ƒ = \{Distributions\ with\ density\ \ f(xI\theta)\}, \theta \in \Theta\ \subseteq R^d$

Given Ƒ satisfies model identifiability condition and it holds that:

$f(xI\theta_1) \neq f(xI\theta_2)$

Then, the joint probability density function of the random sample $X_1, X_2, \dots, X_n$ is given by

$$\prod_{i=1}^{n} f(x_i I\theta_X) \tag{3.1}$$

The Maximum likelihood estimator $\hat{\theta}$ of the parameter θ_X is the point from Θ that maximises the joint density evaluated at the observed values of $X_1, X_2, \dots, X_n$.

The likelihood function of the parameter θ in the model distribution Ƒ is defined as:

$$L(\theta) = \prod_{i=1}^{n} f(x_i I\theta) \tag{3.2}$$

Taking logarithm on both sides, we obtain the log-likelihood function of the parameter space in the model Ƒ as defined as:

$$l(\theta) = \log L(\theta) = \sum_{i=1}^{n} \log f(x_i I\theta) \tag{3.3}$$

The maximum likelihood estimator (MLE) of the parameter space

$$\hat{\theta} = \arg Max\ L(\theta);\ \ , \theta \in \Theta$$

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Since logarithm is strictly increasing, $L(\theta)$ and $l(\theta)$ attain their maximum at the same point hence the estimate that maximises the likelihood function also maximises the log-likelihood function.

The MLE is usually determined by the differentiation of the log-likelihood function.

Consider a random vector whose score function is given as:

$$U(\theta I X_i) = \frac{\partial}{\partial\theta}(\log f(X_i I\theta)) \tag{3.4}$$

And the corresponding score statistics for the parameter θ in the model Ƒ is:

$$U_n(\theta I X_i) = \sum_{i=1}^{n} U(\theta I X_i) = \sum_{i=1}^{n} \frac{\partial}{\partial\theta}(\log f(X_i I\theta)) \tag{3.5}$$

While ith contribution to the information matrix is defined as:

$$I(\theta I X_i) = \frac{\partial U(\theta I X_i)}{\partial\theta^T} = -\frac{\partial^2}{\partial\theta\partial\theta^T}(\log f(X_i I\theta)) \tag{3.6}$$

So, the observed information matrix is given as:

$$I_n(\theta I X_i) = -\frac{1}{n}\frac{\partial}{\partial\theta^T} U_n(\theta I X_i) = \frac{1}{n}\sum_{i=1}^{n} I(\theta I X_i) \tag{3.7}$$

Also, the expected (Fisher) information matrix is defined as:

$$I(\theta) = E\big(I(\theta I X_i)\big) = -E\left(\frac{\partial^2}{\partial\theta\partial\theta^T}(\log f(X_i I\theta))\right) \tag{3.8}$$

If set Θ is open, the MLE $\hat{\theta}_n$ solves the system of equation:

$$U_n(\hat{\theta}_n I X_i) = \sum_{i=1}^{n} \frac{\partial}{\partial\theta}(\log f(X_i I\hat{\theta}_n)) = 0 \tag{3.9}$$

These optimality equations are otherwise known as likelihood equations.

3.3 Model Selection and Evaluation Criteria

The scientific community today is confronted (on regular basis) with the question of the best model that fits occurrence. The quest for an answer to this question motivates many model experts to keep developing new stochastic models and scientifically determining the "best fit" model among the lot. In order not to be trapped with a mad throng of models and new

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models, statistical determination of best-fit model among models becomes very relevance. Hence, it comes as no surprise that many approaches to dealing with this key issue have been proposed over the years. There are many schools of thought about which criteria are very appropriate. Among them, both Bayesian and Frequentist gain popularity on the matter with methods such as cross-validation, stepwise; backward and forward selection procedures, exhaustive search, test for the nested model, AIC, BIC, and so on. Some of the above-mentioned methods are algorithms for selecting a "good" model while others are criteria for judging the quality of a model. Among such criteria for judging the quality of a model are AIC and BIC (MacKay, 1992).

In judging the quality of a model, statisticians use methods that are coherent and general enough to handle a wide range of problems bearing in mind the likelihood principle and principle of parsimony. This is what the Bayesian school of thought proposed (Key, Pericchi, and Smith,1999; MacKay, 1992). Due to the popularity and usability of Bayesian methods of model selection, scholars (Bernardo and Rueda, 2002) dived into model selection as a problem in Bayesian hypothesis testing and failed to reject the Bayesian criterion. This is an indication that the Bayesian factor in model selection stood the test of time.

Among many reviewers who dived into a discussion on Bayes Factor in model selection is Kass and Raftery (1997). This discussion of the Bayes Factor has led to the general acceptability of two criteria used for model selection. These criteria are the Bayesian Information Criterion (BIC) and the Akaike Information Criterion (AIC).

AIC is a criterion that compares the quality of a set of statistical models. It does this by taking the models and ranking them from best to worst. The "best" model among the considered models is the one that neither over-fits nor under-fits that data. AIC chooses the best model among considered models. By implication, if all the models considered are very good, AIC will select the best among them. The converse is also true. That is why it is very important to run the test of significance on any selected model.

$$AIC = -2(loglikelihood) + 2K \quad (3.10)$$

Where K is the number of parameters considered in the model.

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Log-likelihood value measures model suitability.

The first summand (-2loglikelihood) expresses the goodness-of-it of the model while the second summand is a penalty term for increasing the number of parameters in the model. The summands in the AIC equation provide enough ground to harmonise accuracy and parsimony to aid the selection of the best among models.

In conclusion, there is a plethora of approaches in the scientific/statistical community in selecting the best among models. Among this plethora of approaches, Bayesian and classical statisticians' proposals proved to be superior. It is in light of this argument that this study adopts Bayesian-classical criteria in judging the best model among models.

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CHAPTER 4

THEORETICAL RESULTS

4.0 Introduction

This chapter presents the theoretical results of the proposed distributions and their statistical properties. Specifically, the chapter discusses the statistical properties of Exponentiated Janardan distribution, Kumaraswamy Janardan distribution, and Exponentiated Kumaraswamy Janardan distribution.

4.1 Exponentiated Janardan Distribution

In this section, a three-parameter continuous distribution for a non-negative random variable is introduced. It is named "Exponentiated Janardan". This new distribution has the Janardan distribution as the baseline distribution. Visualisation of the behavior of the PDF, hazard rate, and survival function are presented. Also, some statistical properties are established.

For a random variable that follows Janardan Distribution (JD) with parameters φ, ρ ; the probability density function is given as:

$$X \sim JD(x, \varphi, \rho) = f(x, \varphi, \rho).$$

$$g(x, \varphi, \rho) = \frac{\rho^2}{\varphi(\rho+\varphi^2)} . (1+\varphi x)\, e^{-\frac{\rho}{\varphi}x} \;;\; x, \varphi, \rho > 0 \qquad (4.1)$$

and the corresponding cumulative density function (CDF) is established as:

$$G(x, \varphi, \rho) = 1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\, e^{-\frac{\rho}{\varphi}x} ;\; x, \varphi, \rho > 0 \qquad (4.2)$$

The expectation of Janardan distribution is also established as:

$$E(X) = \mu_X = \frac{\varphi(\rho+2\varphi^2)}{\rho(\rho+\varphi^2)} \qquad (4.3)$$

Also, the second, third, and fourth moments are appropriately established.

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$$\mu_2 = 2\left[\frac{\varphi}{\rho}\right]^2 \frac{(\rho+3\varphi^2)}{(\rho+\varphi^2)}\ ;\ \mu_3 = 6\left[\frac{\varphi}{\rho}\right]^3 \frac{(\rho+4\varphi^2)}{(\rho+\varphi^2)}\ ;\ \mu_4 = 24\left[\frac{\varphi}{\rho}\right]^4 \frac{(\rho+5\varphi^2)}{(\rho+\varphi^2)}$$

By generalisation, the rth moment is given as

$$\mu_r = r!\left[\frac{\varphi}{\rho}\right]^r \left[1 + \frac{r\varphi^2}{\rho+\varphi^2}\right]; r = 1,2,3,\ldots \tag{4.4}$$

The mode of the distribution is established as:

Differentiating the pdf and setting the result to zero, the extrema of the Janardan distribution becomes:

$$x = \frac{\varphi^2-\rho}{\varphi\rho}.$$

This value, the $f(x)$ is maximum only if $\varphi > \rho$. But in a situation where $\varphi < \rho$ the $f(x)$ is decreasing in X. Therefore, the mode of Janardan distribution is established to be:

$$mode = \begin{cases} \frac{\varphi^2-\rho}{\varphi\rho}, \varphi > \rho \\ 0, elsewhere \end{cases} \tag{4.5}$$

Hazard function $h(x)$ of the distribution is also established as:

$$h(x) = \frac{\rho^2(1+\varphi x)}{\varphi(\rho+\varphi^2)+\rho\varphi^2 x} \tag{4.6}$$

Janardan probability distribution had been tried and tested to many lifetime datasets and proven to be a more relaxed and better fit than the Lindley distribution.

4.1.1 Derived functions of Exponentiated Janardan Distribution

The cumulative density function for exponential generator

$$F(x) = [G(x)]^\gamma \tag{4.7}$$

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Inserting equation (4.2) into (4.7), we derived the cumulative density function for Exponentiated Janardan (EJ) as:

$$F(x\,) = \left[1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\; e^{-\frac{\rho}{\varphi}x}\right]^{\gamma}$$

Hence, a random variable that conforms to Exponentiated Janardan distribution,

$X \sim EJ(x, \varphi, \rho, \gamma\,)$ has the CDF as:

$$F(x, \varphi, \rho, \gamma\,) = \left[1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\; e^{-\frac{\rho}{\varphi}x}\right]^{\gamma} \qquad (4.8)$$

In order to check the legitimacy of the CDF, the CDF plot of exponentiated Janardan distribution is presented as follows:

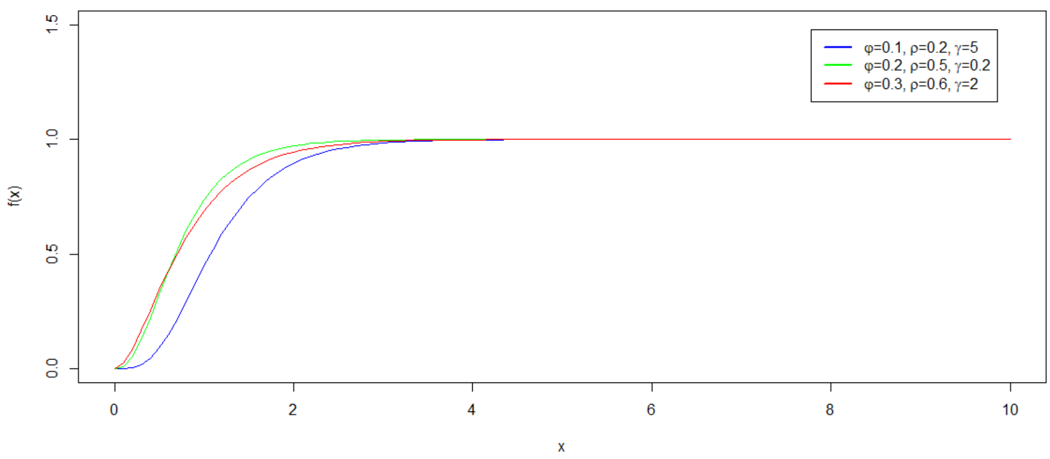

Figure 4.1 Behaviour of the CDF of exponentiated Janardan distribution

Using the chain rule, the CDF (equation 4.8) is differentiated to arrive at the pdf of Exponentiated Janardan distribution as:

Let

$$A = 1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\; e^{-\frac{\rho}{\varphi}x}$$

$$Y = F(x, \varphi, \rho, \gamma\,)$$

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$$Y = [A]^{\gamma}$$

$$\frac{\partial Y}{\partial A} = \gamma[A]^{\gamma-1}$$

$$\frac{\partial A}{\partial x} = \frac{\partial}{\partial x}\left[1 - \frac{\varphi(\rho+\varphi^2)e^{-\frac{\rho}{\varphi}x} + \rho\varphi^2 x e^{-\frac{\rho}{\varphi}x}}{\varphi(\rho+\varphi^2)}\right]$$

$$\frac{\partial A}{\partial x} = \frac{\partial}{\partial x}\left[1 - \frac{\varphi(\rho+\varphi^2)e^{-\frac{\rho}{\varphi}x}}{\varphi(\rho+\varphi^2)} + \frac{\rho\varphi^2 x e^{-\frac{\rho}{\varphi}x}}{\varphi(\rho+\varphi^2)}\right]$$

$$\frac{\partial A}{\partial x} = \left[\frac{\rho(\rho+\varphi^2)e^{-\frac{\rho}{\varphi}x}}{\varphi(\rho+\varphi^2)} + \frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\frac{\partial}{\partial x}\left(xe^{-\frac{\rho}{\varphi}x}\right)\right]$$

Applying the product rule;

$$\frac{\partial A}{\partial x} = \left[\frac{\rho(\rho+\varphi^2)e^{-\frac{\rho}{\varphi}x}}{\varphi(\rho+\varphi^2)} + \frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\left(e^{-\frac{\rho}{\varphi}x} - \frac{\rho}{\varphi}xe^{-\frac{\rho}{\varphi}x}\right)\right]$$

Simplifying:

$$\frac{\partial A}{\partial x} = \left[\frac{\rho(\rho+\varphi^2)e^{-\frac{\rho}{\varphi}x}}{\varphi(\rho+\varphi^2)} - \frac{\rho\varphi^2\, e^{-\frac{\rho}{\varphi}x}}{\varphi(\rho+\varphi^2)} + \frac{\rho^2 x e^{-\frac{\rho}{\varphi}x}}{(\rho+\varphi^2)}\right]$$

$$\frac{\partial A}{\partial x} = \left[\frac{\rho(\rho+\varphi^2)}{\varphi(\rho+\varphi^2)} - \frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)} + \frac{\rho^2\varphi x}{\varphi(\rho+\varphi^2)}\right]e^{-\frac{\rho}{\varphi}x}$$

$$\frac{\partial A}{\partial x} = \left[\frac{\rho^2+\rho\varphi^2}{\varphi(\rho+\varphi^2)} + \frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)} - \frac{\rho^2\varphi x}{\varphi(\rho+\varphi^2)}\right]e^{-\frac{\rho}{\varphi}x}$$

$$\frac{\partial A}{\partial x} = \left[\frac{\rho^2+\rho\varphi^2-\rho\varphi^2+\rho^2\varphi x}{\varphi(\rho+\varphi^2)}\right]e^{-\frac{\rho}{\varphi}x}$$

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$$\frac{\partial A}{\partial x} = \left[\frac{\rho^2 + \rho^2 \varphi x}{\varphi(\rho + \varphi^2)}\right] e^{-\frac{\rho}{\varphi}x}$$

$$\frac{\partial A}{\partial x} = \left[\frac{\rho^2 + \rho^2 \varphi x}{\varphi(\rho + \varphi^2)} e^{-\frac{\rho}{\varphi}x}\right]$$

$$\frac{\partial A}{\partial x} = \left[\frac{\rho^2}{\varphi(\rho + \varphi^2)}(1 + \varphi x) e^{-\frac{\rho}{\varphi}x}\right]$$

Applying the chain rule;

$$\acute{Y} = f(x, \varphi, \rho, \gamma) = \frac{\partial Y}{\partial x} = \frac{\partial Y}{\partial A} \times \frac{\partial A}{\partial x}$$

$$f(x, \varphi, \rho, \gamma) = \gamma \left[1 - \frac{\varphi(\rho + \varphi^2) + \rho\varphi^2 x}{\varphi(\rho + \varphi^2)} e^{-\frac{\rho}{\varphi}x}\right]^{\gamma - 1} \times \left[\frac{\rho^2}{\varphi(\rho + \varphi^2)}(1 + \varphi x) e^{-\frac{\rho}{\varphi}x}\right]$$

Hence, the probability density function of Exponentiated Janardhan distribution is

$$f(x, \varphi, \rho, \gamma) = \gamma \left[\frac{\rho^2}{\varphi(\rho+\varphi^2)}(1 + \varphi x) e^{-\frac{\rho}{\varphi}x}\right]\left[1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)} e^{-\frac{\rho}{\varphi}x}\right]^{\gamma-1} \quad (4.9)$$

The shape of the pdf for some arbitrary values of the parameters of Exponentiated Janardan distribution is presented in figure 4.1. Hence figure 4.1 exhibits the behavior of this proposed distribution.

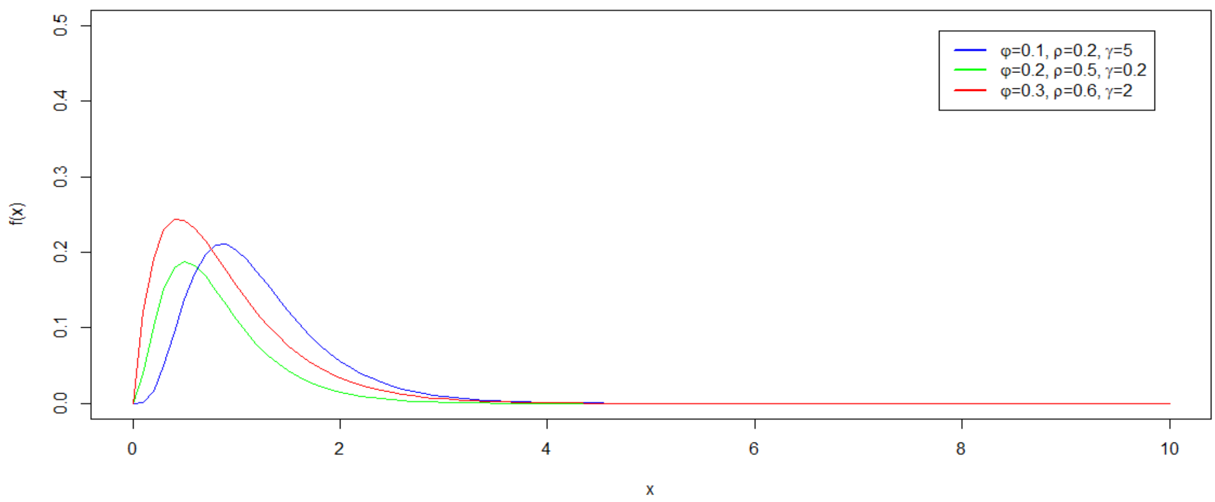

Figure 4.2 Behaviour of PDF of Exponentiated Janardan for some parameters

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Another determinant of the ability of distribution to model life time data is the survival and hazard function of the distribution.

Survival function, $R(x)$, by definition, is given as

$$R(x) = 1 - F(x) \tag{4.10}$$

By implication, the survival function for Exponentiated Janardan distribution is derived as:

$$R(x) = 1 - \left[1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)} e^{-\frac{\rho}{\varphi}x}\right]^{\gamma} \tag{4.11}$$

The behavior of the survival function is pictorially demonstrated in figure 4.2 for some randomly selected values of the parameters of the distribution.

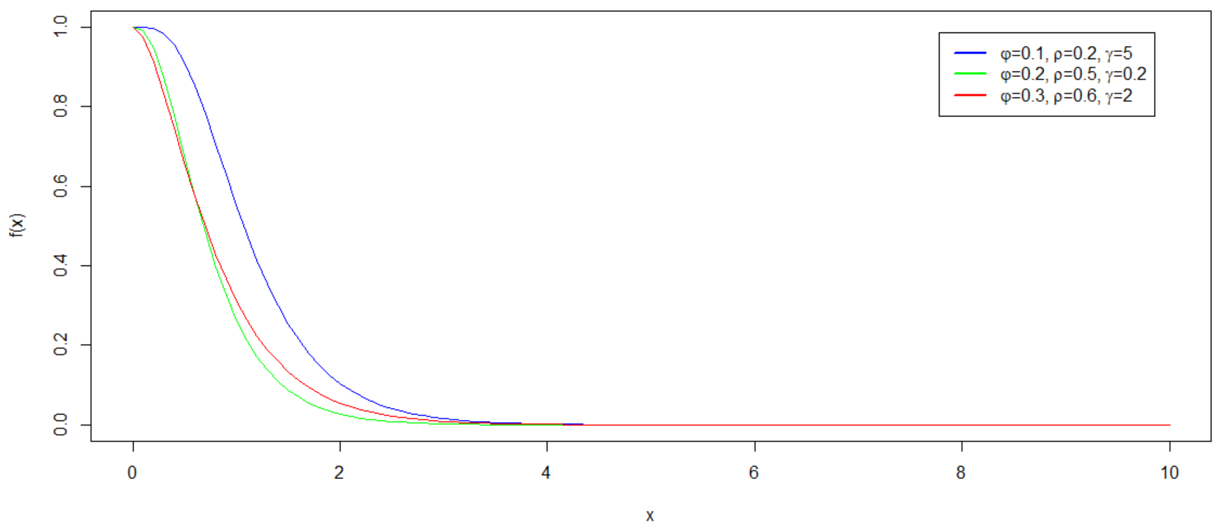

Figure 4.3 Behaviour of Survival function of Exponentiated Janardan for some parameters

Equation (4.12) presents the hazard function of Exponentiated Janardan distribution.

Hazard function, $h(x)$, by definition, is given as:

$$h(x) = \frac{f(x)}{R(x)}$$

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This implies that the Hazard rate function of Exponentiated Janardan distribution is derived as:

$$h(x) = \frac{\gamma\left[\frac{\rho^2}{\varphi(\rho+\varphi^2)}(1+\varphi x)\, e^{-\frac{\rho}{\varphi}x}\right]\left[1-\frac{\varphi(\rho-\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\, e^{-\frac{\rho}{\varphi}x}\right]^{\gamma-1}}{1-\left[1-\frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\, e^{-\frac{\rho}{\varphi}x}\right]^{\gamma}} \qquad (4.12)$$

Furthermore, figure 4.3 demonstrates the behavior of the hazard rate of Exponentiated Janardan distribution for some randomly selected values of the parameters.

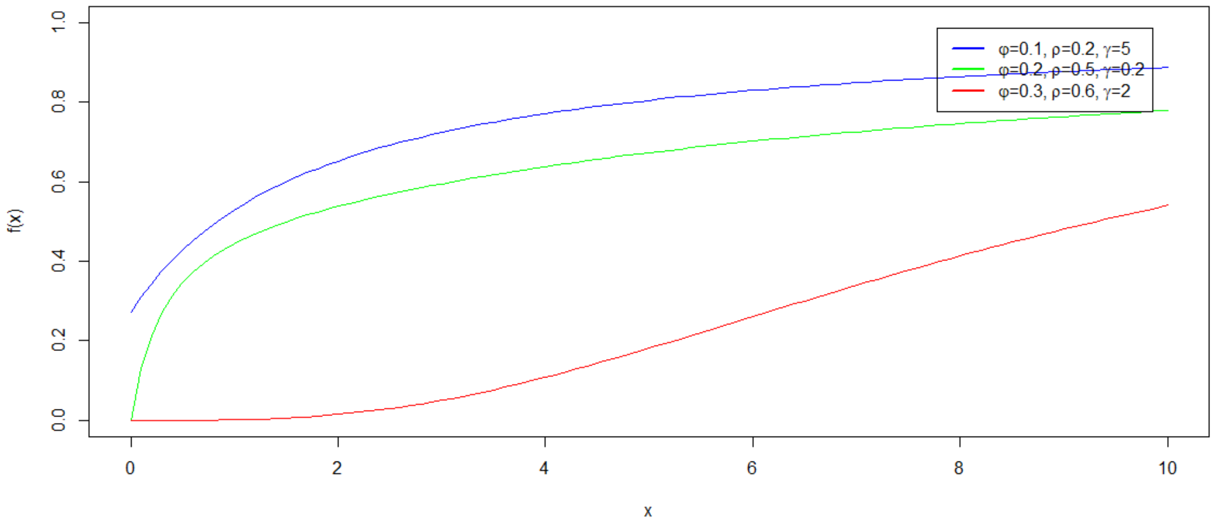

Figure 4.4 Behaviour of Hazard Rate function of Exponentiated Janardan for some parameters

The study further established the role of each of the parameters for Exponentiated Janardan distribution. This is done by varying one parameter while keeping others constant and the result is pictorially displayed in figures 4.4 through figure 4.6.

It is conspicuously clear from figure 4.4 that the parameter gamma, γ, is a location parameter. This gives credence to the proposed distribution's ability to model lifetime data even if the data is transformed or shifted. Even if the data set is shifted to the right, say, x_o+ X (Where X is a random variable emanating from an unknown probability distribution), Exponentiated Janardan would be able to model without a change of shape or form. More information can be gotten about the impact of location parameters from Stone (1975).

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Figure. 4.4 indicates that the varying numeric value of γ only changes the location of the curve without a change of shape while maintaining f(x) at the same value.

Furthermore, rho (ρ) has been determined and illustrated to be the scale parameter of the proposed Exponentiated Janardan distribution. Figure 4.5 reveals that a change in the numeric value of rho (holding all other parameters constant) results in a change in the spread of distribution without a change in the shape of the distribution. This effect is a behavior of the scale parameter of a distribution. Scale parameter controls the variability in the dataset that most lifetime data exhibit.

The functional impact of φ is also determined and illustrated in Figure. 4.6. This is done by keeping all other parameters constant except φ. It is clear from figure 4.6 that, as the value of φ changes, the shape of the distribution also changes. Hence, φ, is playing the role of shape parameter in Exponentiated Janardan distribution.

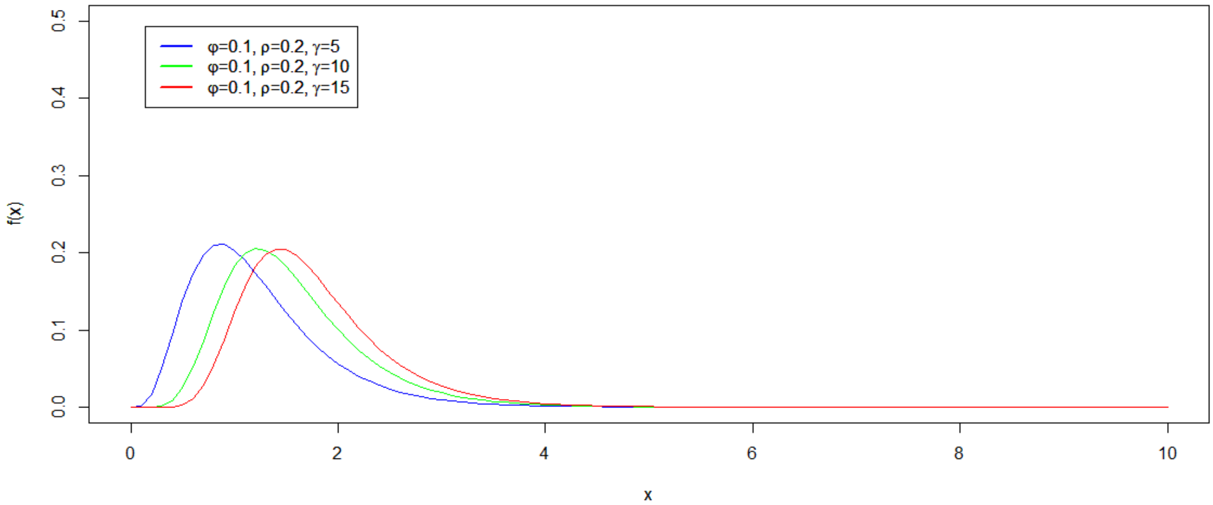

Figure 4.5 Demonstrating the effect of “gamma” on behaviour of pdf of EJ

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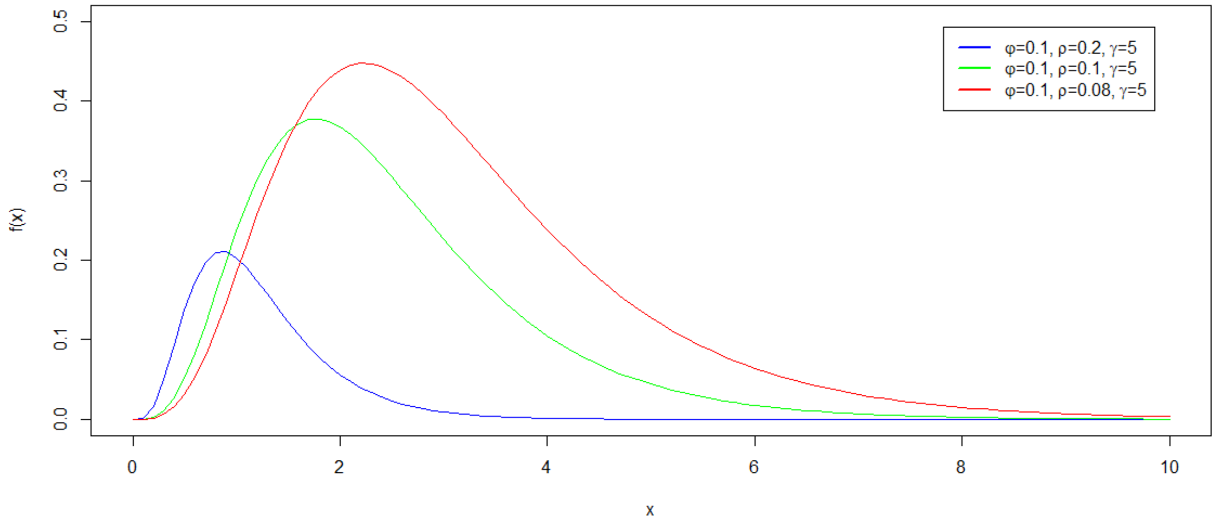

Figure 4.6 Demonstrating the effect of "rho" on behaviour of pdf of EJ

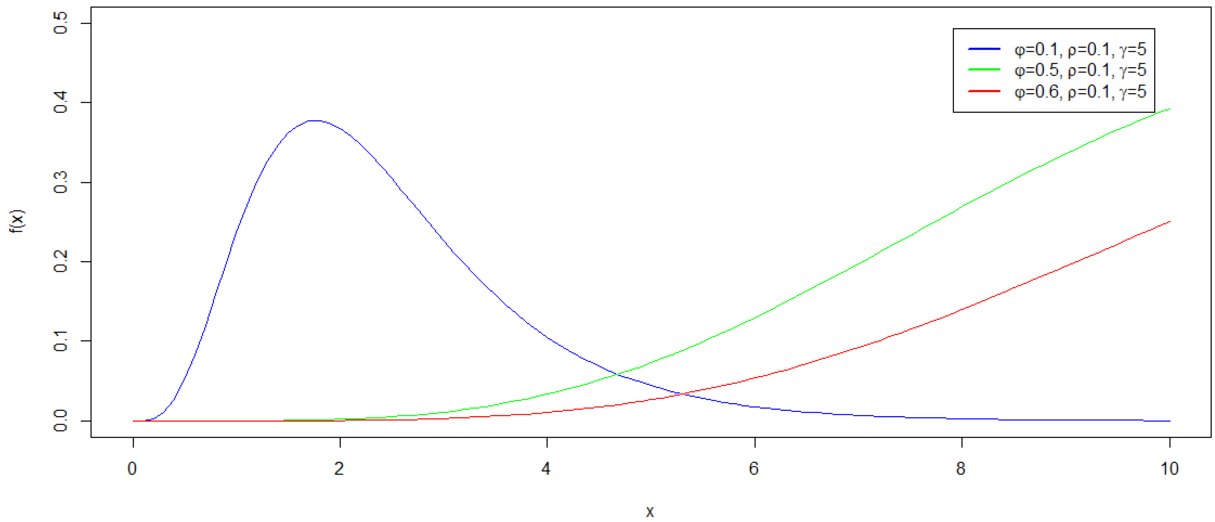

Figure 4.7 Demonstrating the effect of φ on behaviour of pdf of EJ

4.1.2 Statistical Properties of Exponentiated Janardan Distribution

This subsection introduces some statistical properties of Exponentiated Janardan distribution. Some of the statistical properties established are moment about the origin and maximum likelihood estimator of the parameters.

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Moment about the origin

Simplifying (4.9), we obtain

$$f(x,\varphi,\rho,\gamma) = -\frac{\rho\gamma}{\varphi}\left[\frac{-\rho\varphi}{(\rho+\varphi^2)}\right]^{\gamma}\left[\frac{1}{\varphi}x^{\gamma-1}e^{-\frac{\rho\gamma}{\varphi}x} + x^{\gamma}e^{-\frac{\rho\gamma}{\varphi}x}\right] \tag{4.13}$$

Hence, for a random variable "X" that follows exponentiated Janardan distribution with parameters $\varphi, \rho \text{ and } \gamma$ has a probability density function as:

$$f(x,\varphi,\rho,\gamma) = \begin{cases} -\frac{\rho\gamma}{\varphi}\left[\frac{-\rho\varphi}{(\rho+\varphi^2)}\right]^{\gamma}\left[\frac{1}{\varphi}x^{\gamma-1}e^{-\frac{\rho\gamma}{\varphi}x} + x^{\gamma}e^{-\frac{\rho\gamma}{\varphi}x}\right]; & x,\varphi,\rho,\gamma > 0 \\ 0 & Elsewhere \end{cases}$$

$$E(X) = \int_0^\infty x\ f(x)\,dx \tag{4.13a}$$

$$E(X) = -\frac{\rho\gamma}{\varphi}\left[\frac{-\rho\varphi}{(\rho+\varphi^2)}\right]^{\gamma} \times \int_0^\infty x\left[\frac{1}{\varphi}x^{\gamma-1}e^{-\frac{\rho\gamma}{\varphi}x} + x^{\gamma}e^{-\frac{\rho\gamma}{\varphi}x}\right]dx \tag{4.13b}$$

$$E(X) = -\frac{\rho\gamma}{\varphi}\left[\frac{-\rho\varphi}{(\rho+\varphi^2)}\right]^{\gamma} \times \left[\int_0^\infty \frac{1}{\varphi}x^{\gamma}e^{-\frac{\rho\gamma}{\varphi}x} + \int_0^\infty x^{\gamma+1}e^{-\frac{\rho\gamma}{\varphi}x}\right]dx$$

$$E(X) = -\frac{\rho\gamma}{\varphi}\left[\frac{-\rho\varphi}{(\rho+\varphi^2)}\right]^{\gamma} \times \left[\frac{1}{\varphi}\int_0^\infty x^{\gamma}e^{-\frac{\rho\gamma}{\varphi}x} + \int_0^\infty x^{\gamma+1}e^{-\frac{\rho\gamma}{\varphi}x}\right]dx$$

Using gamma transformation:

$$Gamma(\alpha,\beta) = \int_0^\infty x^{\alpha-1}e^{-\frac{1}{\beta}x}dx = \lceil\alpha \times \beta^{\alpha} = (\alpha-1)!\,\beta^{\alpha} \tag{4.13c}$$

the expected value of the variable is as follows:

$$\therefore E(X) = -\frac{\rho\gamma}{\varphi}\left[\frac{-\rho\varphi}{(\rho+\varphi^2)}\right]^{\gamma} \times \left[\frac{1}{\varphi}(\gamma)!\left(\frac{\varphi}{\rho\gamma}\right)^{\gamma+1} + (\gamma+1)!\left(\frac{\varphi}{\rho\gamma}\right)^{\gamma+2}\right] \tag{4.14}$$

In general, the rth moment about the origin is established as follows:

$$E(X^r) = \int_0^\infty x^r f(x)\,dx \tag{4.14a}$$

$$E(X^r) = -\frac{\rho\gamma}{\varphi}\left[\frac{-\rho\varphi}{(\rho+\varphi^2)}\right]^{\gamma}\int_0^\infty\left[\frac{1}{\varphi}x^{\gamma+r-1}e^{-\frac{\rho\gamma}{\varphi}x} + x^{\gamma+r}e^{-\frac{\rho\gamma}{\varphi}x}\right]dx \tag{4.14b}$$

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$$E(X^r) = -\frac{\rho\gamma}{\varphi}\left[\frac{-\rho\varphi}{(\rho+\varphi^2)}\right]^{\gamma}\left[\frac{1}{\varphi}(\gamma+r)!\left(\frac{\varphi}{\rho\gamma}\right)^{\gamma+r} + (\gamma+r+1)!\left(\frac{\varphi}{\rho\gamma}\right)^{\gamma+r+1}\right]$$

$$E(X^r) = -\frac{\rho\gamma}{\varphi}\left(\frac{\varphi}{\rho\gamma}\right)^{1}\left[\frac{-\rho\varphi}{(\rho+\varphi^2)}\right]^{\gamma}\left[\frac{1}{\varphi}(\gamma+r)!\left(\frac{\varphi}{\rho\gamma}\right)^{\gamma+r} + (\gamma+r+1)!\left(\frac{\varphi}{\rho\gamma}\right)^{\gamma+r+1}\right]$$

$$E(X^r) = -\left[\frac{-\rho\varphi}{(\rho+\varphi^2)}\right]^{\gamma}\left[\frac{1}{\varphi}(\gamma+r)!\left(\frac{\varphi}{\rho\gamma}\right)^{\gamma+r} + (\gamma+r+1)!\left(\frac{\varphi}{\rho\gamma}\right)^{\gamma+r+1}\right]$$

$$E(X^r) = -\left[\frac{-\rho\varphi}{(\rho+\varphi^2)}\right]^{\gamma}\left[\frac{1}{\varphi} + (\gamma+r+1)\left(\frac{\varphi}{\rho\gamma}\right)\right](\gamma+r)!\left(\frac{\varphi}{\rho\gamma}\right)^{\gamma+r} \tag{4.14c}$$

When r = 1

$$E(X) = -\left[\frac{-\rho\varphi}{(\rho+\varphi^2)}\right]^{\gamma}\left[\frac{1}{\varphi} + (\gamma+2)\left(\frac{\varphi}{\rho\gamma}\right)\right](\gamma+1)!\left(\frac{\varphi}{\rho\gamma}\right)^{\gamma+1} \tag{4.14d}$$

When r = 2

$$E(X^2) = -\left[\frac{-\rho\varphi}{(\rho+\varphi^2)}\right]^{\gamma}\left[\frac{1}{\varphi} + (\gamma+3)\left(\frac{\varphi}{\rho\gamma}\right)\right](\gamma+2)!\left(\frac{\varphi}{\rho\gamma}\right)^{\gamma+2} \tag{4.14e}$$

When r = 3

$$E(X^3) = -\left[\frac{-\rho\varphi}{(\rho+\varphi^2)}\right]^{\gamma}\left[\frac{1}{\varphi} + (\gamma+4)\left(\frac{\varphi}{\rho\gamma}\right)\right](\gamma+3)!\left(\frac{\varphi}{\rho\gamma}\right)^{\gamma+3} \tag{4.14e}$$

When r = 4

$$E(X^4) = -\left[\frac{-\rho\varphi}{(\rho+\varphi^2)}\right]^{\gamma}\left[\frac{1}{\varphi} + (\gamma+5)\left(\frac{\varphi}{\rho\gamma}\right)\right](\gamma+4)!\left(\frac{\varphi}{\rho\gamma}\right)^{\gamma+4} \tag{4.14f}$$

$$Var(X) = E(X^2) - (E(X))^2 \tag{4.14g}$$

$$Var(X) = -\left[\frac{-\rho\varphi}{(\rho+\varphi^2)}\right]^{\gamma}\left[\frac{1}{\varphi} + (\gamma+3)\left(\frac{\varphi}{\rho\gamma}\right)\right](\gamma+2)!\left(\frac{\varphi}{\rho\gamma}\right)^{\gamma+2}$$

$$-\left(\left[\frac{-\rho\varphi}{(\rho+\varphi^2)}\right]^{\gamma}\left[\frac{1}{\varphi} + (\gamma+2)\left(\frac{\varphi}{\rho\gamma}\right)\right](\gamma+1)!\left(\frac{\varphi}{\rho\gamma}\right)^{\gamma+1}\right)^2 \tag{4.14h}$$

$$Var(X) = -\left[\frac{-\rho\varphi}{(\rho+\varphi^2)}\right]^{\gamma}\left[\frac{1}{\varphi}(\gamma+2)!\left(\frac{\varphi}{\rho\gamma}\right)^{\gamma+2} + (\gamma+3)!\left(\frac{\varphi}{\rho\gamma}\right)^{\gamma+3}\right] - \left(-\left[\frac{-\rho\varphi}{(\rho+\varphi^2)}\right]^{\gamma}\left[\frac{1}{\varphi}(\gamma+1)!\left(\frac{\varphi}{\rho\gamma}\right)^{\gamma+1} + (\gamma+2)!\left(\frac{\varphi}{\rho\gamma}\right)^{\gamma+2}\right]\right)^2$$

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$$Var(X) = -\left[\frac{-\rho\varphi}{(\rho+\varphi^2)}\right]^{\gamma}\left[\frac{1}{\varphi}(\gamma+2)!\left(\frac{\varphi}{\rho\gamma}\right)^{\gamma+2} + (\gamma+3)!\left(\frac{\varphi}{\rho\gamma}\right)^{\gamma+3}\right] - \left[\frac{-\rho\varphi}{(\rho+\varphi^2)}\right]^{2\gamma}\left(\left[\frac{1}{\varphi}(\gamma+1)!\left(\frac{\varphi}{\rho\gamma}\right)^{\gamma+1} + (\gamma+2)!\left(\frac{\varphi}{\rho\gamma}\right)^{\gamma+2}\right]\right)^2$$

$$Var(X) = -\left[\frac{-\rho\varphi}{(\rho+\varphi^2)}\right]^{\gamma}\left[\frac{1}{\varphi}(\gamma+2)!\left(\frac{\varphi}{\rho\gamma}\right)^{\gamma+2} + (\gamma+3)!\left(\frac{\varphi}{\rho\gamma}\right)^{\gamma+3}\right] - \left[\frac{-\rho\varphi}{(\rho+\varphi^2)}\right]^{2\gamma}\left(\left[\frac{1}{\varphi}(\gamma+1)!\left(\frac{\varphi}{\rho\gamma}\right)^{\gamma+1} + (\gamma+2)!\left(\frac{\varphi}{\rho\gamma}\right)^{\gamma+2}\right]\right)^2 \quad (4.14i)$$

Maximum Likelihood Estimation

For simplicity, the pdf is re-written as

$$f(x,\varphi,\rho,\gamma) = \frac{\gamma\rho^2\left[1-\frac{\left(\rho\varphi^2 x+\varphi(\rho+\varphi^2)\right)e^{-\frac{\rho x}{\varphi}}}{\varphi(\rho+\varphi^2)}\right]^{\gamma-1}(\varphi x+1)e^{-\frac{\rho x}{\varphi}}}{\varphi(\rho+\varphi^2)} \quad (4.15)$$

From the foregoing statement, the likelihood function is obtained as:

$$L = \prod_{i=1}^{n} f(x_i,\varphi,\rho,\gamma) \quad (4.16a)$$

$$L(\varphi,\rho,\gamma) = \prod_{i=1}^{n}\frac{\gamma\rho^2\left[1-\frac{\left(\rho\varphi^2 x+\varphi(\rho+\varphi^2)\right)e^{-\frac{\rho x}{\varphi}}}{\varphi(\rho+\varphi^2)}\right]^{\gamma-1}(\varphi x+1)e^{-\frac{\rho x}{\varphi}}}{\varphi(\rho+\varphi^2)} \quad (4.16b)$$

$$L(\varphi,\rho,\gamma) = \left(\frac{\gamma\rho^2}{\varphi(\rho+\varphi^2)}\right)^{n}\prod_{i=1}^{n}\left[1-\frac{\left(\rho\varphi^2 x_i+\varphi(\rho+\varphi^2)\right)e^{-\frac{\rho x_i}{\varphi}}}{\varphi(\rho+\varphi^2)}\right]^{\gamma-1}(\varphi x_i+1)e^{-\frac{\rho x_i}{\varphi}} \quad (4.17a)$$

The log-likelihood function is obtained by subjecting the likelihood function to a natural logarithm as:

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$$\log(\mathrm{L}(\varphi,\rho,\gamma)) = \log\left(\left(\frac{\gamma\rho^2}{\varphi(\rho+\varphi^2)}\right)^n \prod_{i=1}^{n}\left[1-\frac{\left(\rho\varphi^2 x_i+\varphi(\rho+\varphi^2)\right)e^{-\frac{\rho x_i}{\varphi}}}{\varphi(\rho+\varphi^2)}\right]^{\gamma-1}(\varphi x_i+1)e^{-\frac{\rho x_i}{\varphi}}\right)$$

(4.17b)

Evaluation of log-likelihood function yields:

$$\log\big(\mathrm{L}(\varphi,\rho,\gamma)\big) = n\log\left(\frac{\gamma\rho^2}{\varphi(\rho+\varphi^2)}\right) + \sum_{i=1}^{n}\log\left(\left[1-\frac{\left(\rho\varphi^2 x_i+\varphi(\rho+\varphi^2)\right)e^{-\frac{\rho x_i}{\varphi}}}{\varphi(\rho+\varphi^2)}\right]^{\gamma-1}(\varphi x_i+1)e^{-\frac{\rho x_i}{\varphi}}\right)$$

$$\log\big(\mathrm{L}(\varphi,\rho,\gamma)\big) = n\left[\log(\gamma)+2\log(\rho)-\log(\varphi)-\log\big((\rho+\varphi^2)\big)\right] +$$

$$\sum_{i=1}^{n}\left(\left(log\left[1-\frac{(\rho\varphi^2 x_i+\varphi(\rho+\varphi^2))e^{-\frac{\rho x_i}{\varphi}}}{\varphi(\rho+\varphi^2)}\right]^{\gamma-1} + log(\varphi x_i+1)e^{-\frac{\rho x_i}{\varphi}}\right)\right)$$

$$\log\big(\mathrm{L}(\varphi,\rho,\gamma)\big) = n\left[\log(\gamma)+2\log(\rho)-\log(\varphi)-\log\big((\rho+\varphi^2)\big)\right] +$$

$$\sum_{i=1}^{n}\left(\left(log\left[1-\frac{\left(\rho\varphi^2 x_i+\varphi(\rho+\varphi^2)\right)e^{-\frac{\rho x_i}{\varphi}}}{\varphi(\rho+\varphi^2)}\right]^{\gamma-1} + log(\varphi x_i+1)e^{-\frac{\rho x_i}{\varphi}}\right)\right) \qquad (4.17c)$$

$$\log\big(\mathrm{L}(\varphi,\rho,\gamma)\big) = n\left[\log(\gamma)+2\log(\rho)-\log(\varphi)-\log\big((\rho+\varphi^2)\big)\right]$$

$$+(\gamma-1)\sum_{i=1}^{n}\left(\left(log\left[1-\frac{\left(\rho\varphi^2 x_i+\varphi(\rho+\varphi^2)\right)e^{-\frac{\rho x_i}{\varphi}}}{\varphi(\rho+\varphi^2)}\right]\right)\right) + \sum_{i=1}^{n}\left(\left(log(\varphi x_i+1)e^{-\frac{\rho x_i}{\varphi}}\right)\right)$$

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$$\log\big(L(\varphi,\rho,\gamma)\big) = n\left[\log(\gamma) + 2\log(\rho) - \log(\varphi) - \log\big((\rho+\varphi^2)\big)\right] + (\gamma - 1)\sum_{i=1}^{n}\left(\log(\varphi) + \left(\log\big(\rho\varphi x_i + (\rho+\varphi^2)\big)\right) - \frac{\rho x_i}{\varphi} - \left(\log(\varphi) - \log(\rho+\varphi^2)\right)\right) + \sum_{i=1}^{n}\left(\left(log(\varphi x_i + 1)\right) - \frac{\rho x_i}{\varphi}\right)$$

$$\log\big(L(\varphi,\rho,\gamma)\big) = n\left[\log(\gamma) + 2\log(\rho) - \log(\varphi) - \log\big((\rho+\varphi^2)\big)\right] + (\gamma - 1)[\log(\varphi) - \log(\varphi) - \log(\rho+\varphi^2)]\sum_{i=1}^{n}\left(\left(\log\big(\rho\varphi x_i + (\rho+\varphi^2)\big)\right)\right) - \frac{\rho}{\varphi}\sum_{i=1}^{n}(x_i) + \sum_{i=1}^{n} log(\varphi x_i + 1) - \frac{\rho}{\varphi}\sum_{i=1}^{n}(x_i)$$

$$\log\big(L(\varphi,\rho,\gamma)\big) = \log(\gamma) + 2\log(\rho) - \log(\varphi) - \log(\rho+\varphi^2) + (\gamma - 1)[\log(\varphi) - \log(\varphi) - \log(\rho+\varphi^2)]\sum_{i=1}^{n}\left(\left(\log\big(\rho\varphi x_i + (\rho+\varphi^2)\big)\right)\right) - \frac{\rho}{n\varphi}\sum_{i=1}^{n}(x_i) + \sum_{i=1}^{n} log(\varphi x_i + 1) - \frac{\rho}{n\varphi}\sum_{i=1}^{n}(x_i)$$

$$\log\big(L(\varphi,\rho,\gamma)\big) = \log(\gamma) + 2\log(\rho) - \log(\varphi) - \log(\rho+\varphi^2) + (\gamma - 1)[\log(\varphi) - \log(\varphi) - \log(\rho+\varphi^2)]\sum_{i=1}^{n}\left(\left(\log\big(\rho\varphi x_i + (\rho+\varphi^2)\big)\right)\right) - \frac{\rho}{\varphi}\bar{x} + \sum_{i=1}^{n} log(\varphi x_i + 1) - \frac{\rho}{\varphi}\bar{x} \qquad (4.18)$$

Furthermore, the maximum likelihood estimates $\hat{\varphi}, \hat{\rho}, \hat{\gamma}$ of φ, ρ, γ are represented as solutions to the respective optimality equations.

The equation maximiser with respect to φ is

$$\frac{\partial}{\partial\varphi}\log\big(L(\varphi,\rho,\gamma)\big) = 0 \qquad (4.18a)$$

$$\frac{\partial}{\partial\varphi}\log\big(L(\varphi,\rho,\gamma)\big) = -\frac{1}{\varphi} - \frac{2\varphi}{\rho+\varphi^2} + (\gamma - 1)\left[-\frac{2\varphi}{\rho+\varphi^2}\right]\sum_{i=1}^{n}\left(\frac{\rho x_i + 2\varphi}{(\rho\varphi x_i + (\rho+\varphi^2))}\right) + 2\frac{\rho}{\varphi^2}\bar{x} + \sum_{i}^{n}\frac{x_i}{\varphi x_i + 1} = 0 \qquad (4.18b)$$

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The equation maximiser with respect to ρ is

$$\frac{\partial}{\partial \rho}\log\big(L(\varphi,\rho,\gamma)\big) = 0 \tag{4.18c}$$

$$\frac{2}{\rho} - \frac{1}{\rho+\varphi^2} + (\gamma - 1)\left[-\frac{1}{\rho+\varphi^2}\right]\sum_{i=1}^{n}\left(\frac{\varphi x_i + 1}{\rho\varphi x_i + (\rho+\varphi^2)}\right) - 2\frac{\bar{x}}{\varphi} = 0 \tag{4.18d}$$

The equation maximiser with respect to γ is

$$\frac{\partial}{\partial \gamma}\log\big(L(\varphi,\rho,\gamma)\big) = 0 \tag{4.18e}$$

$$\log\big(L(\varphi,\rho,\gamma)\big) = \frac{1}{\gamma} + [-\log(\rho+\varphi^2)] = 0 \tag{4.18f}$$

$$\frac{1}{\gamma} = \log(\rho+\varphi^2)$$

$$\hat{\gamma} = \frac{1}{\log(\hat{\rho}+\hat{\varphi}^2)} \tag{4.18g}$$

Quantile Function for Exponentiated Janardan Distribution

This subsection presents the quantile function of Exponentiated Janardan distribution. According to Chen (2000), the quantile function is very useful in the computation of various characteristics such as percentile, median, quantiles, skewness, and kurtosis.

Mathematically, the quantile function is the inverse of the CDF. A continuous random variable 'X' which is strictly monotonic, a quantile function is expressed mathematically as follows:

Let $u = F(x); 0 < u < 1$

$$u = \left[1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\, e^{-\frac{\rho}{\varphi}x}\right]^{\gamma} \tag{4.18h}$$

Making x the subject of the relation as follows:

$$\left[1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\, e^{-\frac{\rho}{\varphi}x}\right]^{\gamma} = u$$

$$1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\, e^{-\frac{\rho}{\varphi}x} = \sqrt[\gamma]{u}$$

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$$1 - \sqrt[\gamma]{u} \;=\; \frac{\varphi(\rho + \varphi^2) + \rho\varphi^2 x}{\varphi(\rho + \varphi^2)}\, e^{-\frac{\rho}{\varphi}x}$$

$$\left(1 - \sqrt[\gamma]{u}\right)\varphi(\rho + \varphi^2) \;= \varphi(\rho + \varphi^2)e^{-\frac{\rho}{\varphi}x} + \;\rho\varphi^2 x\, e^{-\frac{\rho}{\varphi}x} \tag{4.18i}$$

Taking *ln* on both sides as follows:

$$ln\left(1 - \sqrt[\gamma]{u}\right) + \ln(\varphi) + ln(\rho + \varphi^2) \;= \ln(\varphi) + ln(\rho + \varphi^2) \;- \frac{\rho}{\varphi}x + \ln(\,\rho\varphi^2) + \ln(x) - \frac{\rho}{\varphi}x \tag{4.18j}$$

$$ln\left(1 - \sqrt[\gamma]{u}\right) \;=\; -\frac{\rho}{\varphi}x + \ln(\,\rho\varphi^2) + \ln(x) - \frac{\rho}{\varphi}x$$

$$ln\left(1 - \sqrt[\gamma]{u}\right) - \;\ln(\,\rho\varphi^2) \;=\; -\frac{\rho}{\varphi}x + \ln(x) - \frac{\rho}{\varphi}x$$

$$ln\left(\frac{\left(1 - \sqrt[\gamma]{u}\right)}{\rho\varphi^2}\right) \;=\; -\frac{\rho}{\varphi}x + \ln(x) - \frac{\rho}{\varphi}x$$

$$ln\left(\frac{\left(1 - \sqrt[\gamma]{u}\right)}{\rho\varphi^2}\right) \;=\; \ln(x) - \frac{2\rho}{\varphi}x \tag{4.18k}$$

The quantile function of Exponentiated Janardan distribution does not have the exact solution (a close form), hence quantile function can be approximated using numerical optimisation.

Table 4.1 presents the quantile values of EJ for some monotonic values of $u \in (0,1)$ for arbitrary parameter values. It can be seen that as the values of ‘u’ monotonically increase towards unitary, the quantile values increase.

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Table 4.1 Iterated Quantile values for some arbitrary parameter values

Q(u)	$\varphi = 20, \rho = 9$ and $\gamma = 3$	$\varphi = 6, \rho = 2$ and $\gamma = 5$
Q(0.1)	7.486456	11.44227
Q(0.2)	8.236381	12.55206
Q(0.25)	8.568819	13.02998
Q(0.4)	9.5345	14.38805
Q(0.6)	10.96573	16.35231
Q(0.75)	12.40909	18.30435
Q(0.8)	13.05235	19.16951
Q(0.9)	14.95424	21.71939

4.2 Kumaraswamy Janardan Distribution

In this section, a four-parameter distribution named as Kumaraswamy-Janardan (KJ) distribution is presented. Cumulative density function, probability density function, hazard rate, survival rate functions, and their respective pictorial presentation are established in this section. Also, the maximum likelihood estimator as well as a moment about the origin are established for K-J.

4.2.1 Mathematical Derivations of Kumaraswamy-Janardan Distribution

From literature, Shanker et al. (2013) established the Cumulative density function and density function of Janardan distribution respectively as:

$$G(x, \varphi, \rho) = 1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\, e^{-\frac{\rho}{\varphi}x};\ x, \varphi, \rho\ > 0 \tag{4.19}$$

$$g(x, \varphi, \rho\,) = \frac{\rho^2}{\varphi(\rho+\varphi^2)}.(1 + \varphi x)\, e^{-\frac{\rho}{\varphi}x}\ ;\ x, \varphi, \rho\ > 0 \tag{4.20}$$

Also, Kumaraswamy (1980) established cumulative and probability density functions of Kumaraswamy distribution respectively as:

$$F(x) = 1 - (1 - x^{\alpha})^{\beta} \tag{4.21}$$

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And

$$f(x) = \alpha\beta x^{\alpha-1}(1 - x^{\alpha})^{\beta-1} \tag{4.22}$$

Kumaraswamy distribution was later established as Kumaraswamy generalised by Cordeiro and Castro (2011) with CDF and PDF respectively as:

$$F(x) = 1 - (1 - G(x)^{\alpha})^{\beta};\;\; 0 < x < 1;\; \alpha, \beta > 0 \tag{4.23}$$

In order to establish the cumulative density function for the Kumaraswamy-Janardan distribution, we employed the function of functions approach by substituting equation (4.19) into equation (4.23).

$$F(x) = \begin{cases} 1 - \left(1 - \left(1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\; e^{-\frac{\rho}{\varphi}x}\right)^{\alpha}\right)^{\beta} & ; x, \varphi, \rho, \alpha, \beta > 0 \\ 0; & elsewhere \end{cases} \tag{4.24}$$

Where φ *is scale parameter and shape parameters being* ρ, α *and* β

Differentiating equation (4.24);

$$f(x) = \left[\frac{\alpha\beta\rho\varphi}{(\rho+\varphi^2)}\left(\frac{\rho x}{\varphi} - 1\right)e^{-\frac{\rho}{\varphi}x}\right] \times \left[\left(1 - \left(1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\; e^{-\frac{\rho}{\varphi}x}\right)^{\alpha}\right)^{\beta-1}\right] \times$$

$$\left[\left(1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\; e^{-\frac{\rho}{\varphi}x}\right)^{\alpha-1}\right] \tag{4.25}$$

Having established CDF and PDF, as indicated in equation (4.24) and equation (4.25) respectively, the study established the mathematical function of the survival function and hazard function as follows:

Survival function, $R(x)$, by definition, is given as

$$R(x) = 1 - F(x)$$

This implies that the survival function for Kumaraswamy Janardan distribution is derived as:

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$$R(x) = 1 - \left[1 - \left(1 - \left(1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\ e^{-\frac{\rho}{\varphi}x}\right)^{\alpha}\right)^{\beta}\right] \quad (4.26)$$

Hazard function, $H(x)$, by definition, is given as:

$$H(x) = \frac{f(x)}{R(x)}$$

This implies that the Hazard rate function of Kumaraswamy Janardan distribution is derived as:

$$H(x) = \left[\left[\frac{\alpha\beta\rho\varphi}{(\rho+\varphi^2)}\left(\frac{\rho x}{\varphi} - 1\right) e^{-\frac{\rho}{\varphi}x}\right] \times \left[\left(1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\ e^{-\frac{\rho}{\varphi}x}\right)^{\alpha-1}\right] \times \left[\left(1 - \left(1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\ e^{-\frac{\rho}{\varphi}x}\right)^{\alpha}\right)^{\beta-1}\right]\right] \div \left[1 - \left[1 - \left(1 - \left(1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\ e^{-\frac{\rho}{\varphi}x}\right)^{\alpha}\right)^{\beta}\right]\right] \quad (4.27)$$

It is important to establish the hazard rate function of K-J distribution because it provides the foundation for planning insurance, and the safety of a system in a wider variety of applications (Mahmoud, 2016)

The K-J distribution is very flexible noticing that the distribution at various parameter values exhibits several renowned distributions as sub-models. For instance;

a) When $\alpha = 1$ a three-parameter Exponentiated Janardan distribution is obtained and its CDF is obtained as follows:

$$F(x) = 1 - \left(- \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\ e^{-\frac{\rho}{\varphi}x}\right)^{\beta} \quad (4.28)$$

b) When $\alpha = \beta = 1$ a two-parameter Janardan distribution is obtained and its CDF is obtained as follows:

$$F(x) = 1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\ e^{-\frac{\rho}{\varphi}x} \quad (4.29)$$

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c) When $\alpha = \beta = \varphi = 1$ a one-parameter Lindley distribution is obtained and its CDF is obtained as follows:

$$F(x) = 1 - \frac{(\rho+1)\rho x}{(\rho+1)} e^{-\rho x} \tag{4.30}$$

d) When $\alpha = \beta = \rho = 1$ a gamma distribution is obtained and its CDF is obtained as follows:

$$F(\varphi) = -\frac{\varphi}{(1+\varphi^2)} x e^{-\frac{1}{\varphi}x} \tag{4.31}$$

4.2.2 Graphical Presentation of Kumaraswamy-Janardan Distribution

This section presents a pictorial analysis of the proposed distribution.

From figure 4.7, it is conspicuous that the Kumaraswamy-Janardan distribution is a unimodal probability distribution. Depending on parameter values, the distribution is depicting the flexibility of modeling datasets that are right skewed or nearly symmetric.

A graphical illustration of the effect of the various parameters on the distribution is presented in Figures 4.8 to figure 4.11. Careful observation of the behavior of the figures reveal that φ is scale parameter while α, ρ and β are the shape parameters. φ being scale parameter controls the variability and scalability in the dataset. The roles of these parameters are demonstrated pictorially.

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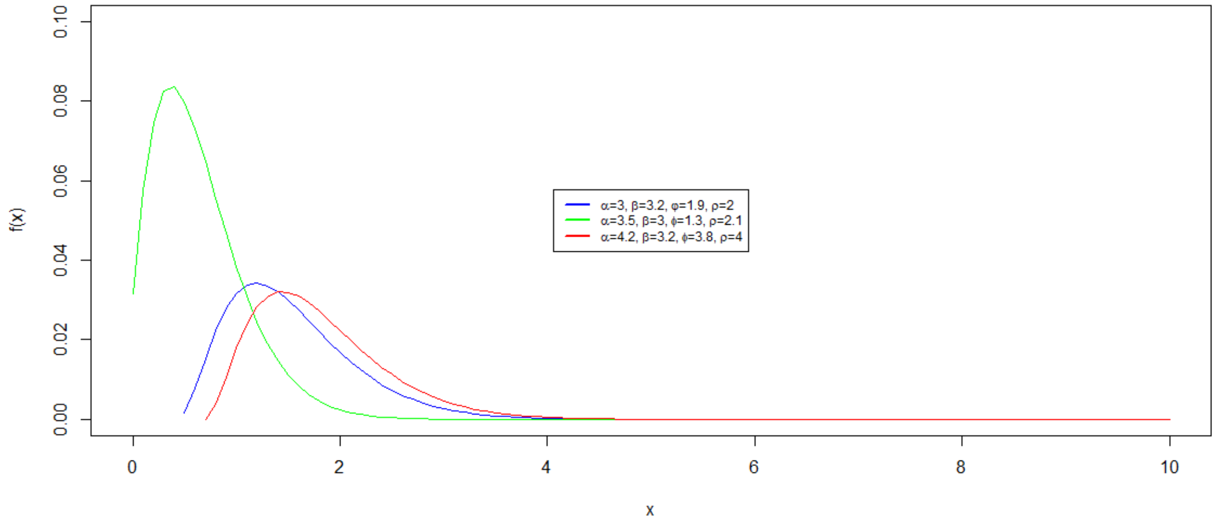

Figure 4.8 Behaviour of pdf of KJ for some parameters

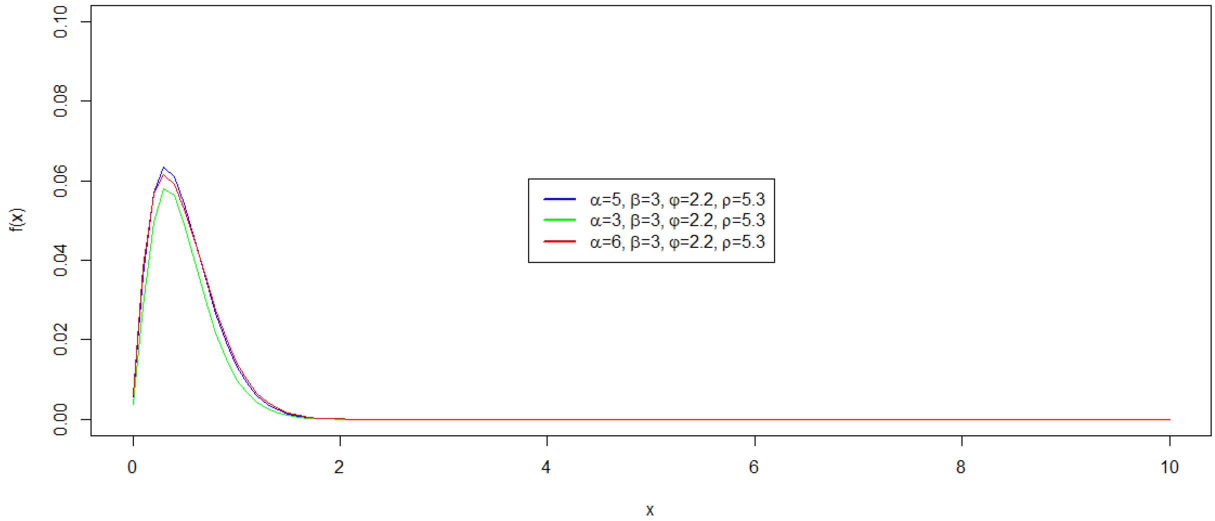

Figure 4.9 Behaviour of pdf of KJ with varying α value

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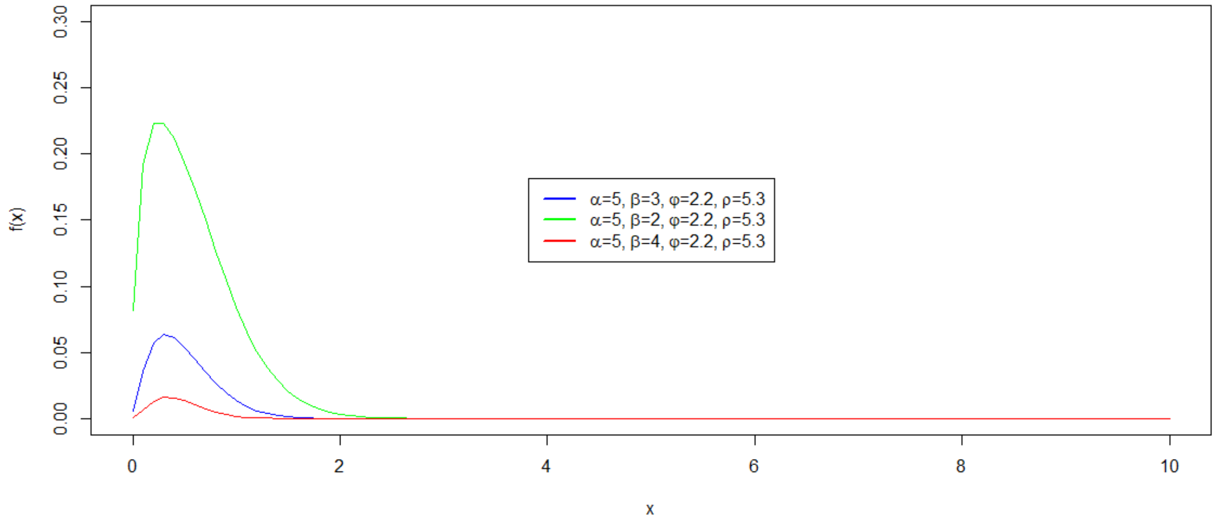

Figure 4.10 Behaviour of pdf of KJ with varying β value

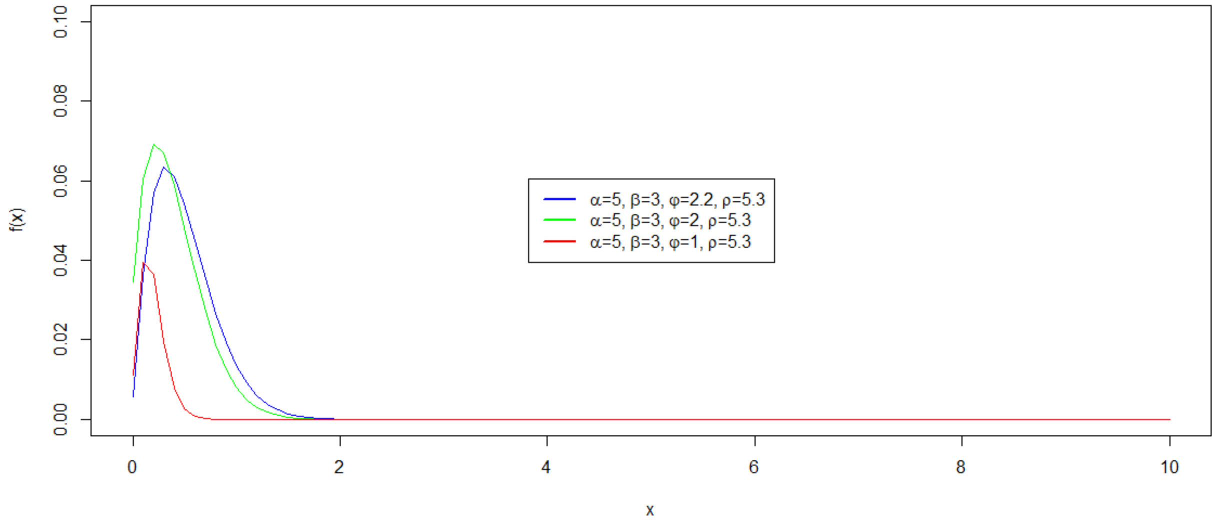

Figure 4.11 Behaviour of pdf of KJ with varying φ value

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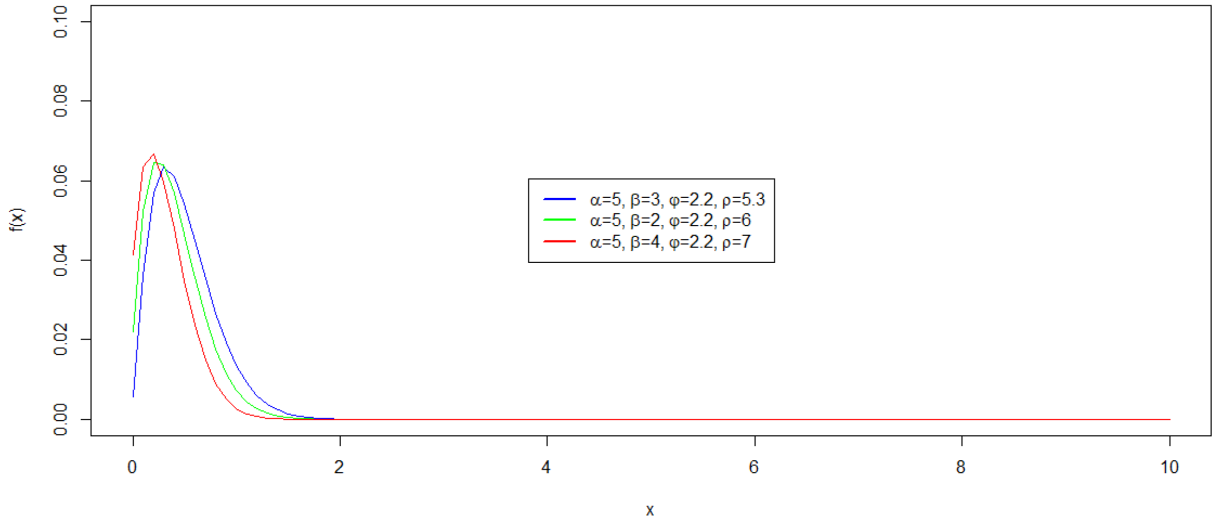

Figure 4.12 Behaviour of pdf of KJ with varying ρ value

From figure 4.8 through figure 4.11, it is clear that "peakness" of the KJ model changes with varying values of α, ρ and β depicting that they shape parameters. Though these three are shape parameters, the magnitude of their impact on shape of the distribution differ. This is indication that their values would differ when numerical optimisation is run. Combining the three in modeling would provide optimised result in controlling skewness and kurtosis. Also, φ being scale parameter controls the variability and scalability in a given dataset.

4.2.3 Linear Representation of Probability Function of Kumaraswamy-Janardan Distribution

Due to complex nature of PDF, determination of statistical properties becomes complex and time consuming. To reduce this complexity, the PDF is transformed as linear representation using binomial series expansion as demonstrated in this section.

Recall equation (4.25) and renamed as follows:

$$f(x) = \left[\frac{\alpha\beta\rho\varphi}{(\rho+\varphi^2)}\left(\frac{\rho x}{\varphi} - 1\right)\right] \times \left[\left(1 - \left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\, x e^{-\frac{\rho}{\varphi}x}\right)^{\alpha}\right)^{\beta-1}\right] \times \left[\left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\, x e^{-\frac{\rho}{\varphi}x}\right)^{\alpha-1}\right] e^{-\frac{\rho}{\varphi}x} \qquad (4.32)$$

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In mathematics, binomial series of $(1+b)^p$ can be written linearly as $\sum_{i=0}^{n}\binom{p}{i} b^i$.

Hence, Binomial series presentation employed in simplification of the pdf of Kumaraswamy distribution is:

$$(1+b)^p = \sum_{i=0}^{n}\binom{p}{i} b^i \tag{4.33a}$$

Applying equation (4.33) to (4.32),

$$f(x) = \sum_{i=0}^{\beta-1}\binom{\beta-1}{i} (-1)^{\alpha i}\left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\, xe^{-\frac{\rho}{\varphi}x}\right)^{\alpha i} \times \left[\left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\, xe^{-\frac{\rho}{\varphi}x}\right)^{\alpha-1}\right] e^{-\frac{\rho}{\varphi}x}$$

Simplifying to separate the variable x from the constants, resulting in:

$$f(x) = \sum_{i=0}^{\beta-1}\binom{\beta-1}{i} (-1)^{\alpha i}\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha i}\left(xe^{-\frac{\rho}{\varphi}x}\right)^{\alpha i} \times \left[\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\, xe^{-\frac{\rho}{\varphi}x}\right)^{\alpha-1}\right] e^{-\frac{\rho}{\varphi}x} \tag{4.33b}$$

$$f(x) = \sum_{i=0}^{\beta-1}\binom{\beta-1}{i} (-1)^{\alpha i}\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha i}(x)^{\alpha i}\left(e^{-\frac{\rho}{\varphi}x}\right)^{\alpha i} \times \left[\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\, xe^{-\frac{\rho}{\varphi}x}\right)^{\alpha-1}\right] e^{-\frac{\rho}{\varphi}x}$$

$$f(x) = \sum_{i=0}^{\beta-1}\binom{\beta-1}{i} (-1)^{\alpha i}\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha i}(x)^{\alpha i}\left(e^{-\alpha i\frac{\rho}{\varphi}x}\right)\left[\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\, xe^{-\frac{\rho}{\varphi}x}\right)^{\alpha-1}\right] e^{-\frac{\rho}{\varphi}x}$$

$$f(x) = \sum_{i=0}^{\beta-1}\binom{\beta-1}{i} (-1)^{\alpha i}\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha i}(x)^{\alpha i}\left[\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\, xe^{-\frac{\rho}{\varphi}x}\right)^{\alpha-1}\right] e^{-\frac{\rho}{\varphi}x}\left(e^{-\alpha i\frac{\rho}{\varphi}x}\right)$$

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$$f(x) = \sum_{i=0}^{\beta-1} \binom{\beta-1}{i} (-1)^{\alpha i} \left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha i} (x)^{\alpha i} \left[\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)} xe^{-\frac{\rho}{\varphi}x}\right)^{\alpha-1}\right] e^{-(1+\alpha i)\frac{\rho}{\varphi}x}$$

$$f(x) = \sum_{i=0}^{\beta-1} \binom{\beta-1}{i} (-1)^{\alpha i} \left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha i} (x)^{\alpha i} \left[\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)} xe^{-\frac{\rho}{\varphi}x}\right)^{\alpha-1}\right] e^{-(1+\alpha i)\frac{\rho}{\varphi}x}$$

$$f(x) = \sum_{i=0}^{\beta-1} \binom{\beta-1}{i} (-1)^{\alpha i} \left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha i} (x)^{\alpha i} \left[\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha-1} (x)^{\alpha-1}\right] \times \left[e^{-(1+\alpha i)\frac{\rho}{\varphi}x} e^{-\frac{\rho}{\varphi}(\alpha-1)x}\right]$$

$$f(x) = \sum_{i=0}^{\beta-1} \binom{\beta-1}{i} (-1)^{\alpha i} \left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha i} (x)^{\alpha i} \left[\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha-1} (x)^{\alpha-1}\right] \times \left[e^{-((1+\alpha i)\frac{\rho}{\varphi}x+(\alpha-1)\frac{\rho}{\varphi}x)}\right]$$

$$f(x) = \sum_{i=0}^{\beta-1} \binom{\beta-1}{i} (-1)^{\alpha i} \left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha i} (x)^{\alpha i} \left[\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha-1} (x)^{\alpha-1}\right] \times \left[e^{-((1+\alpha i)+(\alpha-1))\frac{\rho}{\varphi}x}\right]$$

$$f(x) = \sum_{i=0}^{\beta-1} \binom{\beta-1}{i} (-1)^{\alpha i} \left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha i} (x)^{\alpha i} \left[\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha-1} (x)^{\alpha-1}\right] e^{-((\alpha i)+(\alpha))\frac{\rho}{\varphi}x}$$

$$f(x) = \sum_{i=0}^{\beta-1} \binom{\beta-1}{i} (-1)^{\alpha i} \left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha i} (x)^{\alpha i} \left[\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha-1} (x)^{\alpha-1}\right] e^{-(\alpha i+\alpha)\frac{\rho}{\varphi}x}$$

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$$f(x) = \sum_{i=0}^{\beta-1} \binom{\beta-1}{i} (-1)^{\alpha i} \left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)} \right)^{\alpha i} \left[\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)} \right)^{\alpha-1} \right] (x)^{\alpha i+\alpha-1} e^{-(\alpha i+\alpha)\frac{\rho}{\varphi}x}$$

(4.33c)

Let

$$A = \sum_{i=0}^{\beta-1} \binom{\beta-1}{i} (-1)^{\alpha i} \left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)} \right)^{\alpha i} \left[\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)} \right)^{\alpha-1} \right]$$

Then

$$f(x) = A \; x^{(\alpha i+\alpha-1)} \; e^{-(\alpha i+\alpha)\frac{\rho}{\varphi}x} \qquad (4.34)$$

4.2.4 Statistical Properties of Kumaraswamy-Janardan Distribution

In this section, we introduce some statistical properties of the Kumaraswamy-Janardan distribution. Some of the statistical properties established are moment about the origin and maximum likelihood estimates of the parameters.

Moment and Moment Generating Function

For a random variable X that follows Kumaraswamy-Janardan distribution has pdf as:

$$f(x) = \left[\frac{\alpha\beta\rho\varphi}{(\rho+\varphi^2)} \left(\frac{\rho x}{\varphi} - 1 \right) \right] \times \left[\left(1 - \left(- \frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)} \, x e^{-\frac{\rho}{\varphi}x} \right)^{\alpha} \right)^{\beta-1} \right] \left[\left(- \frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)} \, x e^{-\frac{\rho}{\varphi}x} \right)^{\alpha-1} \right] e^{-\frac{\rho}{\varphi}x}$$

With its linear representation as:

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$$f(x) = A\ x^{(\alpha i+\alpha-1)}\ e^{-(\alpha i+\alpha)\frac{\rho}{\varphi}x}$$

The r[th] moment about the origin is defined as:

$$E(X^r) = \int_0^\infty x^{\,r} f(x)\,dx \qquad (4.35)$$

Substituting (4.34) into (4.35), results in:

$$E(X^r) = A\int_0^\infty x^{\,r}\ \ x^{(\alpha i+\alpha-1)}\ e^{-(\alpha i+\alpha)\frac{\rho}{\varphi}x}\ dx \qquad (4.36a)$$

$$E(X^r) = A\int_0^\infty x^{(\alpha i+\alpha+r)-1}\ e^{-(\alpha i+\alpha)\frac{\rho}{\varphi}x}\ dx \qquad (4.36b)$$

Using gamma transformation;

$$Gamma(\alpha,\beta) = \int_0^\infty x^{\alpha-1}e^{-\frac{1}{\beta}x}\,dx = \lceil\alpha \times \beta^\alpha = (\alpha-1)!\,\beta^\alpha$$

$$E(X^r) = A\left[(\alpha i+\alpha+r)!\left(\frac{\varphi}{(\alpha i+\alpha)\rho}\right)^{\alpha i+\alpha+r+1}\right] \qquad (4.37)$$

$$E(X^r) = \begin{cases} A\left[(\alpha i+\alpha+r)!\left(\dfrac{\varphi}{(\alpha i+\alpha)\rho}\right)^{\alpha i+\alpha+r+1}\right], & r = 1,2,3,\ldots \\ 0, & elsewhere \end{cases}$$

From (4.37), it follows, therefore, that:

$$E(X) = A\left[(\alpha i+\alpha+1)!\left(\frac{\varphi}{(\alpha i+\alpha)\rho}\right)^{\alpha i+\alpha+2}\right] \qquad (4.37a)$$

$$E(X^2) = A\left[(\alpha i+\alpha+2)!\left(\frac{\varphi}{(\alpha i+\alpha)\rho}\right)^{\alpha i+\alpha+3}\right] \qquad (4.37b)$$

$$E(X^3) = A\left[(\alpha i+\alpha+3)!\left(\frac{\varphi}{(\alpha i+\alpha)\rho}\right)^{\alpha i+\alpha+4}\right] \qquad (4.37c)$$

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$$E(X^4) = A\left[(\alpha i + \alpha + 4)!\left(\frac{\varphi}{(\alpha i+\alpha)\rho}\right)^{\alpha i+\alpha+5}\right] \quad (4.37d)$$

The moment generating function of KJ distribution is given by

$$M_x(t) = \int_0^\infty e^{tx} f(x)\, dx \quad (4.38)$$

Using the fact that $e^{tx} = \sum_{r=0}^{\infty}\frac{(tx)^r}{r!}$, to obtain:

$$M_x(t) = \int_0^\infty \sum_{r=0}^{\infty}\frac{(tx)^r}{r!} f(x)\, dx \quad (4.39)$$

$$M_x(t) = \int_0^\infty \sum_{r=0}^{\infty}\frac{(t)^r}{r!} x^r f(x)\, dx$$

$$M_x(t) = \sum_{r=0}^{\infty}\frac{(t)^r}{r!}\int_0^\infty x^r f(x)\, dx$$

$$M_x(t) = \sum_{r=0}^{\infty}\frac{(t)^r}{r!} E(X^r) \quad (4.40)$$

Maximum Likelihood Estimation (MLE)

In probability, Maximum Likelihood Estimation is used to estimate parameters of probability distribution given some observed sample data. According to Rossi (2018), by maximising the likelihood function of the probability model, the sample data is most probable. The estimated point in the parameter space that maximises the likelihood function is called the maximum likelihood estimate. Any given set of observations is a sample from an unknown population, and MLE is to help make inferences about the population that is most likely to have generated the sample (Myung, 2003).

It is against this background that this section presents the maximum likelihood estimation of the Kumaraswamy-Janardan Distribution.

Intuitively, given parameter space $\emptyset = [\alpha \;\; \beta \;\; \rho \;\; \varphi]^T$, MLE $L(\emptyset, x)$ is given as:

$$L(\emptyset, x) = \prod_{i=1}^{n} f(x_i/\emptyset) \quad (4.41)$$

Plugging (4.32) into (4.41), to obtain

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$$L(\emptyset, x) = \prod_1^n \left\{ \begin{array}{c} \left[\frac{\alpha\beta\rho\varphi}{(\rho+\varphi^2)}\right]\left[\frac{\rho x}{\varphi} - 1\right] \times \left[\left(1 - \left(1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\; e^{-\frac{\rho}{\varphi}x}\right)^{\alpha}\right)^{\beta-1}\right] \\ \times \left(1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\; e^{-\frac{\rho}{\varphi}x}\right)^{\alpha-1} \; e^{-\frac{\rho}{\varphi}x} \end{array} \right\}$$

$$L(\emptyset, x) = \left[\frac{\alpha\beta\rho\varphi}{(\rho+\varphi^2)}\right]^n \prod_1^n \left\{ \begin{array}{c} \left[\frac{\rho x}{\varphi} - 1\right] \times \left[\left(1 - \left(1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\; e^{-\frac{\rho}{\varphi}x}\right)^{\alpha}\right)^{\beta-1}\right] \\ \times \left(1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\; e^{-\frac{\rho}{\varphi}x}\right)^{\alpha-1} \; e^{-\frac{\rho}{\varphi}x} \end{array} \right\} \qquad (4.42)$$

Taking "ln" on both sides to arrive at log-likelihood (l) and it is given as:

$$l = n\left(\ln\frac{\alpha\beta\rho\varphi}{(\rho+\varphi^2)}\right) + \sum_{i=1}^n ln\left\{\left[\frac{\rho x}{\varphi} - 1\right] \times \left[\left(1 - \left(1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\; e^{-\frac{\rho}{\varphi}x}\right)^{\alpha}\right)^{\beta-1}\right] \times \left(1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\; e^{-\frac{\rho}{\varphi}x}\right)^{\alpha-1} \; e^{-\frac{\rho}{\varphi}x}\right\} \qquad (4.42a)$$

$$l = n\left(\ln\frac{\alpha\beta\rho\varphi}{(\rho+\varphi^2)}\right) + \sum_{i=1}^n \left\{ \begin{array}{c} ln\left[\frac{\rho x}{\varphi} - 1\right] + (\beta-1)ln\left(1 - \left(1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\; e^{-\frac{\rho}{\varphi}x}\right)^{\alpha}\right) \\ +(\alpha-1)\ln\left(1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\; e^{-\frac{\rho}{\varphi}x}\right) \; -\frac{\rho}{\varphi}x \end{array} \right\}$$

$$l = n\left(\ln\frac{\alpha\beta\rho\varphi}{(\rho+\varphi^2)}\right) + \sum_{i=1}^n \left\{ \begin{array}{c} ln\left[\frac{\rho x - \varphi}{\varphi}\right] + (\beta-1)ln\left(1 - \left(1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\; e^{-\frac{\rho}{\varphi}x}\right)^{\alpha}\right) \\ +(\alpha-1)\ln\left(1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\; e^{-\frac{\rho}{\varphi}x}\right) \; -\frac{\rho}{\varphi}x \end{array} \right\}$$

$$l = n[\ln(\alpha) + \ln(\beta) + \ln(\rho) + \ln(\varphi) - \ln(\rho+\varphi^2)] + \sum_{i=1}^n \left\{ \begin{array}{c} \ln(\rho x - \varphi) - \ln(\varphi) \\ +(\beta-1)ln\left(1 - \left(1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\; e^{-\frac{\rho}{\varphi}x}\right)^{\alpha}\right) \\ +(\alpha-1)\ln\left(1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\; e^{-\frac{\rho}{\varphi}x}\right) \; -\frac{\rho}{\varphi}x \end{array} \right\} \qquad (4.42b)$$

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$$l = n[\ln(\alpha) + \ln(\beta) + \ln(\rho) + \ln(\varphi) - \ln(\rho + \varphi^2)] + \sum_1^n \ln(\rho x - \varphi) - n\ln(\varphi)$$

$$+(\beta - 1)\sum_1^n \ln\left(1 - \left(1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\, e^{-\frac{\rho}{\varphi}x}\right)^{\alpha}\right) + (\alpha - 1)\sum_1^n \ln\left(1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\, e^{-\frac{\rho}{\varphi}x}\right) - \frac{\rho n\bar{x}}{\varphi} \quad (4.43)$$

The objective here is to find the values of the respective parameters in the model that maximise the likelihood function over the parameter space, $\emptyset$. This objective can be achieved if the log-likelihood function is partially differentiated over the parameter space ($\frac{\partial l}{\partial \alpha}, \frac{\partial l}{\partial \beta}, \frac{\partial l}{\partial \rho}$ *and* $\frac{\partial l}{\partial \varphi}$).

The estimated point in the parameter space $(\hat{\varphi}, \hat{\rho}, \hat{\alpha}, \hat{\beta})$ that maximises the likelihood function of KJ distribution is achieved by simultaneously solving the optimality equations:

$$\frac{\partial l}{\partial \alpha} = 0, \quad \frac{\partial l}{\partial \beta} = 0, \quad \frac{\partial l}{\partial \rho} = 0 \quad and \quad \frac{\partial l}{\partial \varphi} = 0$$

The optimality equations are complex to solve analytically but can be solved empirically using an iterative technique like the Simulated annealing algorithm (a package in R-Studio).

4.3 Exponentiated Kumaraswamy Janardan Distribution

The third proposed distribution is Exponentiated Kumaraswamy Janardan (EKJ). This is a five-parameter model obtained through the method of parameterisation. This distribution has Shanker's Janardan model as a baseline.

Recall that Janardan distribution has cumulative density function as in equation (4.2) and renamed as follows:

$$G(x) = 1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\, e^{-\frac{\rho}{\varphi}x} \quad (4.44)$$

Where x is a random variable; and ρ and φ are the parameters to be estimated.

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Also, the cumulative density function for the exponentiated generator as in equation (4.7) is renamed and given as follows:

$$F(x) = [G(x)]^{\gamma} \quad (4.45)$$

The cumulative density function of Kumaraswamy distribution (4.23) is recalled, renamed, and given as follows:

$$F(x) = 1 - (1 - G(x)^{\alpha})^{\beta}; \;\; 0 < x < 1; \; \alpha, \beta > 0 \quad (4.46)$$

4.3.1 Cumulative density Function of Proposed EKJ

The third proposed model is Exponentiated Kumaraswamy Janardan (EKJ). This is a five-parameter model with proposed CDF as follows:

$$F(x) = \left[1 - \left(1 - \left[1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)} e^{-\frac{\rho}{\varphi}x}\right]^{\alpha}\right)^{\beta}\right]^{\gamma} \quad (4.47)$$

$$\varphi, \rho, \alpha, \beta, \gamma, x > 0$$

While x is a random variable, φ *and* ρ are scale parameters, and α, β *and* γ are shape parameters.

<u>Proof</u>

The CDF for Kumaraswamy is given by Cordeiro and Castro (2011) as:

$$F(x) = 1 - (1 - G(x)^{\alpha})^{\beta}$$

This generator has two shape parameters with no scale parameter. This makes Kumaraswamy a good generator to solve limitations in the Janardan model.

Substituting the Janardan CDF,

$G(x) = 1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)} e^{-\frac{\rho}{\varphi}x}$ into Kumaraswamy generator, $F(x) = 1 - (1 - G(x)^{\alpha})^{\beta}$;

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Kumaraswamy-Janardan model is obtained as follows:

$$G(x) = 1 - \left(1 - \left[1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\, e^{-\frac{\rho}{\varphi}x}\right]^{\alpha}\right)^{\beta} \quad (4.48)$$

Substituting CDF of Kumaraswamy-Janardan as in equation (4.48) into Exponentiated generator as in equation (4.45):

$$F(x) = [G(x)]^{\gamma},$$

The CDF of Exponentiated Kumaraswamy Janardan is obtained as:

$$F(x) = \left[1 - \left(1 - \left[1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\, e^{-\frac{\rho}{\varphi}x}\right]^{\alpha}\right)^{\beta}\right]^{\gamma}$$

$$\varphi, \rho, \alpha, \beta, \gamma, x > 0$$

Hence the resultant equation (4.47) is the CDF for the proposed five (5) parameters Exponentiated Kumaraswamy distribution. This distribution has three shape parameters and two scale parameters.

4.3.2: Probability Density function of proposed EKJ

From first principle, pdf of any distribution is obtained by differentiating the cdf of the distribution. Same principle holds in this model as well.

This implies that equation (4.47) is differentiated, to obtain pdf of Exponentiated Kumaraswamy Janardan and the result is presented in equation (4.49) as follows:

$$f(x) = \gamma\beta\alpha\left[1 - \left(1 - \left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)} x e^{-\frac{\rho x}{\varphi}}\right)^{\alpha}\right)^{\beta}\right]^{\gamma-1}\left[1 - \left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)} x e^{-\frac{\rho x}{\varphi}}\right)^{\alpha}\right]^{\beta-1}$$

$$\times$$

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$$\left[-\frac{\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}e^{-\frac{\rho x}{\varphi}}\right]^{\alpha-1}\left[\frac{\rho}{\varphi}-\frac{\rho\varphi}{(\rho+\varphi^2)}+\frac{\rho x}{\varphi}\right]e^{-\frac{\rho x}{\varphi}} \quad (4.49)$$

Proof:

Recall equation (4.47)

$$F(x) = \left[1-\left(1-\left[1-\frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\;e^{-\frac{\rho}{\varphi}x}\right]^{\alpha}\right)^{\beta}\right]^{\gamma}$$

Let $A = 1-\left(1-\left[1-\frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\;e^{-\frac{\rho}{\varphi}x}\right]^{\alpha}\right)^{\beta}$

$$F(x) = [A]^{\gamma}$$

Differentiating the function with respect to A as follow:

$$\frac{dF(x)}{dA} = \gamma[A]^{\gamma-1}$$

Let $B = \left(1-\left[1-\frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\;e^{-\frac{\rho}{\varphi}x}\right]^{\alpha}\right)$.

$A = 1-\ (B)^{\beta}$

Differentiating A with respect to B as follows:

$\frac{dA}{dB} = \ -\beta[B]^{\beta-1}$

Let $C = \ 1-\frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\;e^{-\frac{\rho}{\varphi}x}$

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$B = 1 - (C)^{\alpha}$

$\frac{dB}{dC} = -\alpha[C]^{\alpha-1}$

Differentiating C with respect to x as follows:

$$\frac{dC}{dx} = \left[\frac{\rho\varphi(\rho+\varphi^2)}{\varphi^2(\rho+\varphi^2)} - \frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)} + \frac{\rho x}{\varphi}\right] e^{-\frac{\rho}{\varphi}x}$$

Applying chain rule;

$$\frac{dF(x)}{dx} = f(x) = \frac{dF(x)}{dA} \cdot \frac{dA}{dB} \cdot \frac{dB}{dC} \cdot \frac{dC}{dx} \tag{4.50}$$

But

$$\frac{dF(x)}{dA} = \gamma[A]^{\gamma-1}, \frac{dA}{dB} = -\beta[B]^{\beta-1}, \frac{dB}{dC} = -\alpha[C]^{\alpha-1} \text{ and } \frac{dC}{dx} = \left[\frac{\rho\varphi(\rho+\varphi^2)}{\varphi^2(\rho+\varphi^2)} - \frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)} + \frac{\rho x}{\varphi}\right] e^{-\frac{\rho}{\varphi}x}$$

Substituting into equation (4.50) results as follows:

$$f(x) = \gamma[A]^{\gamma-1} . -\beta[B]^{\beta-1} . -\alpha[C]^{\alpha-1} . \left[\frac{\rho\varphi(\rho+\varphi^2)}{\varphi^2(\rho+\varphi^2)} - \frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)} + \frac{\rho x}{\varphi}\right] e^{-\frac{\rho}{\varphi}x} \tag{4.51}$$

Substituting the expression for A, B and C to obtain

$$f(x) = \gamma\left[1 - \left(1 - \left[1 - \frac{\varphi(\rho+\varphi^2) + \rho\varphi^2 x}{\varphi(\rho+\varphi^2)} e^{-\frac{\rho}{\varphi}x}\right]^{\alpha}\right)^{\beta}\right]^{\gamma-1} \times$$

$$-\beta\left[1 - \left[1 - \frac{\varphi(\rho+\varphi^2) + \rho\varphi^2 x}{\varphi(\rho+\varphi^2)} e^{-\frac{\rho}{\varphi}x}\right]^{\alpha}\right]^{\beta-1} \times$$

$$-\alpha\left[1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)} e^{-\frac{\rho}{\varphi}x}\right]^{\alpha-1} \times \left[\frac{\rho\varphi(\rho+\varphi^2)}{\varphi^2(\rho+\varphi^2)} - \frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)} + \frac{\rho x}{\varphi}\right] e^{-\frac{\rho}{\varphi}x} \tag{4.51a}$$

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$$f(x) = \gamma\beta\alpha\left[1-\left(1-\left(1-\frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}e^{-\frac{\rho x}{\varphi}}\right)^{\alpha}\right)^{\beta}\right]^{\gamma-1}\left[1-\left(1-\frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}e^{-\frac{\rho x}{\varphi}}\right)^{\alpha}\right]^{\beta-1}$$

$$\left[1-\frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}e^{-\frac{\rho x}{\varphi}}\right]^{\alpha-1}\left[\frac{\rho\varphi(\rho+\varphi^2)}{\varphi^2(\rho+\varphi^2)}-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}+\frac{\rho x}{\varphi}\right]e^{-\frac{\rho x}{\varphi}}$$

Simplifying

$$f(x) = \gamma\beta\alpha\left[1-\left(1-\left(1-\frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}e^{-\frac{\rho x}{\varphi}}\right)^{\alpha}\right)^{\beta}\right]^{\gamma-1}\left[1-\left(1-\frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}e^{-\frac{\rho x}{\varphi}}\right)^{\alpha}\right]^{\beta-1}\times$$

$$\left[1-\frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}e^{-\frac{\rho x}{\varphi}}\right]^{\alpha-1}\left[\frac{\rho\varphi(\rho+\varphi^2)}{\varphi^2(\rho+\varphi^2)}-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}+\frac{\rho x}{\varphi}\right]e^{-\frac{\rho x}{\varphi}} \quad (4.51b)$$

$$f(x) = \gamma\beta\alpha\left[1-\left(1-\left(1-\frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}e^{-\frac{\rho x}{\varphi}}\right)^{\alpha}\right)^{\beta}\right]^{\gamma-1}\left[1-\left(1-\frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}e^{-\frac{\rho x}{\varphi}}\right)^{\alpha}\right]^{\beta-1}\times$$

$$\left[1-\frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}e^{-\frac{\rho x}{\varphi}}\right]^{\alpha-1}\left[\frac{\rho\varphi(\rho+\varphi^2)}{\varphi^2(\rho+\varphi^2)}-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}+\frac{\rho x}{\varphi}\right]e^{-\frac{\rho x}{\varphi}}$$

$$f(x) = \gamma\beta\alpha\left[1-\left(1-\left(1-\frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}e^{-\frac{\rho x}{\varphi}}\right)^{\alpha}\right)^{\beta}\right]^{\gamma-1}\left[1-\left(1-\frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}e^{-\frac{\rho x}{\varphi}}\right)^{\alpha}\right]^{\beta-1}\times\left[1-\frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}e^{-\frac{\rho x}{\varphi}}\right]^{\alpha-1}\left[\frac{\rho\varphi(\rho+\varphi^2)}{\varphi^2(\rho+\varphi^2)}-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}+\frac{\rho x}{\varphi}\right]e^{-\frac{\rho x}{\varphi}}$$

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$$f(x) = \gamma\beta\alpha\left[1-\left(1-\left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}xe^{-\frac{\rho x}{\varphi}}\right)^{\alpha}\right)^{\beta}\right]^{\gamma-1}\left[1-\left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}xe^{-\frac{\rho x}{\varphi}}\right)^{\alpha}\right]^{\beta-1}$$

$$\times$$

$$\left[-\frac{\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}e^{-\frac{\rho x}{\varphi}}\right]^{\alpha-1}\left[\frac{\rho}{\varphi}-\frac{\rho\varphi}{(\rho+\varphi^2)}+\frac{\rho x}{\varphi}\right]e^{-\frac{\rho x}{\varphi}} \quad (4.51c)$$

Hence, the pdf equation (4.49) is proven.

4.3.3 Linear Representation of Probability Density function of EKJ

Recall equation (4.49):

$$f(x) = \gamma\beta\alpha\left[1-\left(1-\left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}xe^{-\frac{\rho x}{\varphi}}\right)^{\alpha}\right)^{\beta}\right]^{\gamma-1}\left[1-\left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}xe^{-\frac{\rho x}{\varphi}}\right)^{\alpha}\right]^{\beta-1}$$

$$\times$$

$$\left[-\frac{\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}e^{-\frac{\rho x}{\varphi}}\right]^{\alpha-1}\left[\frac{\rho}{\varphi}-\frac{\rho\varphi}{(\rho+\varphi^2)}+\frac{\rho x}{\varphi}\right]e^{-\frac{\rho x}{\varphi}}$$

Also, recall equation (4.33) and renamed as follows:

$$(1+b)^p = \sum_{i=0}^{n}\binom{p}{i} b^i \quad (4.52)$$

Applying equation (4.52) to equation (4.49),

$$f(x) = \gamma\beta\alpha\left[\begin{array}{c}\sum_{i=0}^{n}\binom{\gamma-1}{i}(-1)^{\beta i}\left(1-\left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}xe^{-\frac{\rho x}{\varphi}}\right)^{\alpha}\right)^{\beta^i} \\ \sum_{j=0}^{n}\binom{\beta}{j}(-1)^{\alpha j}\left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}xe^{-\frac{\rho x}{\varphi}}\right)^{\alpha j}\end{array}\right]$$

$$\times\left[\sum_{k=0}^{n}\binom{\beta-1}{k}(-1)^{\alpha k}\left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}xe^{-\frac{\rho x}{\varphi}}\right)^{\alpha k}\right]\times$$

$$\left[-\frac{\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}e^{-\frac{\rho x}{\varphi}}\right]^{\alpha-1}\left[\frac{\rho}{\varphi}-\frac{\rho\varphi}{(\rho+\varphi^2)}+\frac{\rho x}{\varphi}\right]e^{-\frac{\rho x}{\varphi}} \quad (4.52a)$$

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$$f(x) = \gamma\beta\alpha \left[\begin{array}{c} \sum_{i=0}^{n} \binom{\gamma-1}{i} (-1)^{\beta i} \left(1 - \left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)} x e^{-\frac{\rho x}{\varphi}}\right)^{\alpha}\right)^{\beta^i} \\ \sum_{j=0}^{n} \binom{\beta}{j} (-1)^{\alpha j} \left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)} x e^{-\frac{\rho x}{\varphi}}\right)^{\alpha j} \end{array} \right]$$

$$\times \left[\sum_{k=0}^{n} \binom{\beta-1}{k} (-1)^{\alpha k} \left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)} x e^{-\frac{\rho x}{\varphi}}\right)^{\alpha k} \right] \times$$

$$\left[-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)} x e^{-\frac{\rho x}{\varphi}}\right]^{\alpha-1} \left[\frac{\rho(\rho+\varphi^2) - \rho\varphi + \rho(\rho+\varphi^2)}{\varphi(\rho+\varphi^2)} x\right] e^{-\frac{\rho x}{\varphi}}$$

$$f(x) = \gamma\beta\alpha \left[\begin{array}{c} \sum_{i=0}^{n} \binom{\gamma-1}{i} (-1)^{\beta i} \left(1 - \left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)} x e^{-\frac{\rho x}{\varphi}}\right)^{\alpha}\right)^{\beta^i} \\ \sum_{j=0}^{n} \binom{\beta}{j} (-1)^{\alpha j} \left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)} x e^{-\frac{\rho x}{\varphi}}\right)^{\alpha j} \end{array} \right]$$

$$\times \left[\sum_{k=0}^{n} \binom{\beta-1}{k} (-1)^{\alpha k} \left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)} x e^{-\frac{\rho x}{\varphi}}\right)^{\alpha k} \right] \times$$

$$\left[-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)} x e^{-\frac{\rho x}{\varphi}}\right]^{\alpha-1} \left[\frac{\rho(\rho+\varphi^2) - \rho\varphi + \rho(\rho+\varphi^2)}{\varphi(\rho+\varphi^2)} x\right] e^{-\frac{\rho x}{\varphi}} \qquad (4.52b)$$

Simplifying to separate the variable, x, from the constants, as follows:

$$f(x) = \gamma\beta\alpha \left[\begin{array}{c} \sum_{i=0}^{n} \binom{\gamma-1}{i} (-1)^{\beta i} \left(1 - \left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha}\right)^{\beta^i} \left(x e^{-\frac{\rho x}{\varphi}}\right)^{\alpha\beta i} \\ \sum_{j=0}^{n} \binom{\beta}{j} (-1)^{\alpha j} \left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha j} \left(x e^{-\frac{\rho x}{\varphi}}\right)^{\alpha j} \end{array} \right]$$

$$\times \left[\sum_{k=0}^{n} \binom{\beta-1}{k} (-1)^{\alpha k} \left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha k} \left(x e^{-\frac{\rho x}{\varphi}}\right)^{\alpha k} \right] \times$$

$$\left[-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right]^{\alpha-1} \left[x e^{-\frac{\rho x}{\varphi}}\right]^{\alpha-1} \left[\frac{\rho(\rho+\varphi^2) - \rho\varphi + \rho(\rho+\varphi^2)}{\varphi(\rho+\varphi^2)}\right] x e^{-\frac{\rho x}{\varphi}}$$

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$$f(x) = \gamma\beta\alpha \left[\sum_{i=0}^{n} \binom{\gamma-1}{i} (-1)^{\beta i} \left(1 - \left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha}\right)^{\beta^{i}} \sum_{j=0}^{n} \binom{\beta}{j} (-1)^{\alpha j} \left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha j} \right]$$

$$\times \left[\sum_{k=0}^{n} \binom{\beta-1}{k} (-1)^{\alpha k} \left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha k} \right] \times$$

$$\left[-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right]^{\alpha-1} \left[\frac{\rho(\rho+\varphi^2) - \rho\varphi + \rho(\rho+\varphi^2)}{\varphi(\rho+\varphi^2)}\right] \times \left[xe^{-\frac{\rho x}{\varphi}}\right]^{\alpha-1}$$

$$\left(xe^{-\frac{\rho x}{\varphi}}\right)^{\alpha k} \left(xe^{-\frac{\rho x}{\varphi}}\right)^{\alpha j} \left(xe^{-\frac{\rho x}{\varphi}}\right)^{\alpha\beta i} xe^{-\frac{\rho x}{\varphi}} \qquad (4.52c)$$

$$f(x) = \gamma\beta\alpha \left[\sum_{i=0}^{n} \binom{\gamma-1}{i} (-1)^{\beta i} \left(1 - \left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha}\right)^{\beta^{i}} \sum_{j=0}^{n} \binom{\beta}{j} (-1)^{\alpha j} \left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha j} \right]$$

$$\times \left[\sum_{k=0}^{n} \binom{\beta-1}{k} (-1)^{\alpha k} \left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha k} \right] \times$$

$$\left[-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right]^{\alpha-1} \left[\frac{\rho(\rho+\varphi^2) - \rho\varphi + \rho(\rho+\varphi^2)}{\varphi(\rho+\varphi^2)}\right] (x)^{\alpha-1+\alpha k+\alpha j+\alpha\beta i+1} \left(e^{-\frac{\rho x}{\varphi}}\right)^{\alpha k+\alpha-1+\alpha j+\alpha\beta i+1}$$

$$f(x) = \gamma\beta\alpha \left[\sum_{i=0}^{n} \binom{\gamma-1}{i} (-1)^{\beta i} \left(1 - \left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha}\right)^{\beta} \sum_{j=0}^{n} \binom{\beta}{j} (-1)^{\alpha j} \left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha j^{i}} \right] \times$$

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$$\left[\sum_{k=0}^{n}\binom{\beta-1}{k}(-1)^{\alpha k}\left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha k}\right]\left[-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right]^{\alpha-1}\left[\frac{\rho(\rho+\varphi^2)-\rho\varphi+\rho(\rho+\varphi^2)}{\varphi(\rho+\varphi^2)}\right]$$

$\times$

$$\left[(x)^{\alpha-1+\alpha k+\alpha j+\alpha\beta i+1}\left(e^{-\frac{\rho x}{\varphi}}\right)^{\alpha k+\alpha-1+\alpha j+\alpha\beta i+1}\right]$$

$$f(x)=\left[\begin{matrix}\sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n}\binom{\gamma-1}{i}\binom{\beta}{j}\binom{\beta-1}{k}(-1)^{\alpha k}(-1)^{\beta i}(-1)^{\alpha j}\times\\ \left(1-\left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha}\right)^{\beta i}\quad\left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha j}\end{matrix}\right]\times$$

$$\left[-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right]^{\alpha k}\left[-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right]^{\alpha-1}\left[\frac{\rho(\rho+\varphi^2)-\rho\varphi+\rho(\rho+\varphi^2)}{\varphi(\rho+\varphi^2)}(\gamma\beta\alpha)\right]\times$$

$$\left[(x)^{\alpha-1+\alpha k+\alpha j+\alpha\beta i+1}\left(e^{-\frac{\rho x}{\varphi}}\right)^{\alpha k+\alpha-1+\alpha j+\alpha\beta i+1}\right]$$

$$f(x)=\left[\begin{matrix}\sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n}\binom{\gamma-1}{i}\binom{\beta}{j}\binom{\beta-1}{k}(-1)^{\alpha k}(-1)^{\beta i}(-1)^{\alpha j}\times\\ \left(1-\left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha}\right)^{\beta i}\quad\left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha j}\end{matrix}\right]\times$$

$$\left[-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right]^{\alpha k}\left[-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right]^{\alpha-1}\left[\frac{\rho(\rho+\varphi^2)-\rho\varphi+\rho(\rho+\varphi^2)}{\varphi(\rho+\varphi^2)}(\gamma\beta\alpha)\right]\times$$

$$\left[(x)^{\alpha+\alpha k+\alpha j+\alpha\beta i}\left(e^{-\frac{\rho x}{\varphi}}\right)^{\alpha k+\alpha+\alpha j+\alpha\beta i}\right] \tag{4.52d}$$

$$f(x)=$$

$$\left[\sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n}\binom{\gamma-1}{i}\binom{\beta}{j}\binom{\beta-1}{k}(-1)^{\alpha k}(-1)^{\beta i}(-1)^{\alpha j}\left(1-\right.\right.$$

$$\left.\left.\left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha}\right)^{\beta i}\quad\left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha j}\right]\times$$

$$\left[-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right]^{\alpha k}\left[-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right]^{\alpha-1}\left[\frac{\rho(\rho+\varphi^2)-\rho\varphi+\rho(\rho+\varphi^2)}{\varphi(\rho+\varphi^2)}(\gamma\beta\alpha)\right]\times$$

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$$\left[(x)^{\alpha+\alpha k+\alpha j+\alpha\beta i}\left(e^{-\frac{\rho x}{\varphi}}\right)^{\alpha k+\alpha+\alpha j+\alpha\beta i}\right] \quad (4.52e)$$

Let

$$W_{ijk} =$$

$$\Bigg[\sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n}\binom{\gamma-1}{i}\binom{\beta}{j}\binom{\beta-1}{k}(-1)^{\alpha k}(-1)^{\beta i}(-1)^{\alpha j}\Bigg(1-$$

$$\left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha}\Bigg)^{\beta i}\left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha j}\Bigg]\times$$

$$\left[-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right]^{\alpha k}\left[-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right]^{\alpha-1}\left[\frac{\rho(\rho+\varphi^2)-\rho\varphi+\rho(\rho+\varphi^2)}{\varphi(\rho+\varphi^2)}(\gamma\beta\alpha)\right]$$

By implication, the pdf of EKJ is given as:

$$f(x) = W_{ijk}\left[(x)^{\alpha+\alpha k+\alpha j+\alpha\beta i}\left(e^{-\frac{\rho x}{\varphi}}\right)^{\alpha k+\alpha+\alpha j+\alpha\beta i}\right]$$

$$f(x) = W_{ijk}\left[x^{\alpha(k+j+\beta i+1)}\times e^{-\alpha(k+j+\beta i+1)\frac{\rho x}{\varphi}}\right] \quad (4.53)$$

4.3.4 Statistical Properties of EKJ

This section presents some statistical properties of Exponentiated Kumaraswamy Janardan probability distribution. Some of the statistical properties presented are moments and Maximum Likelihood estimates.

Moments and Moment Generating Function of EKJ

According to Spanos (1999), Pafnuty Chebyshev became the first person to introduce moments of random variables. Moments of random variables are quantitative measures that describe the shape of a probability function's graph. The first moment is known as the expected value while the second moment (crude or central) is a pointer to the spread of the distribution. The third and the fourth standardised moments are the distribution's skewness

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and kurtosis. For a distribution whose random variable is on a bounded interval (from 0 to ∞), the collection of all moments uniquely describes the distribution. It is against this background that this section considers all moments of Exponentiated Kumaraswamy Janardan Distribution.

Recall equation (4.53):

$$f(x) = W_{ijk}\left[x^{\alpha(k+j+\beta i+1)} \times e^{-\alpha(k+j+\beta i+1)\frac{\rho x}{\varphi}}\right]$$

Also, the r^{th} raw moment is defined as:

$$E(X^r) = \int_0^\infty x^r f(x)\,dx \tag{4.54}$$

Applying equation (4.53) to equation (4.54) results in:

$$E(X^r) = W_{ijk}\int_0^\infty x^r\left[x^{\alpha(k+j+\beta i+1)} \times e^{-\alpha(k+j+\beta i+1)\frac{\rho x}{\varphi}}\right] dx$$

$$E(X^r) = W_{ijk}\int_0^\infty \left[x^{\alpha(k+j+\beta i+1)+r} \times e^{-\alpha(k+j+\beta i+1)\frac{\rho x}{\varphi}}\right] dx$$

$$E(X^r) = W_{ijk}\int_0^\infty \left[x^{(\alpha k+\alpha j+\alpha\beta i+\alpha+1+r)-1} \times e^{-(\alpha k+\alpha j+\alpha\beta i+\alpha)\frac{\rho x}{\varphi}}\right] dx \tag{4.55}$$

Using gamma transformation on equation (4.55) results in equation (4.56) below.

$$Gamma(\alpha,\beta) = \int_0^\infty x^{\alpha-1}e^{-\frac{1}{\beta}x}dx = \Gamma\alpha \times \beta^\alpha = (\alpha-1)!\,\beta^\alpha$$

$$E(X^r) = W_{ijk}\left[(\alpha k+\alpha j+\alpha\beta i+\alpha+1+r)!\left(\frac{\varphi}{(\alpha k+\alpha j+\alpha\beta i+\alpha)\rho}\right)^{(\alpha k+\alpha j+\alpha\beta i+\alpha+1+r)}\right] \tag{4.56}$$

$$E(X^r) =$$

$$\begin{cases} W_{ijk}\left[(\alpha k+\alpha j+\alpha\beta i+\alpha+1+r)!\left(\dfrac{\varphi}{(\alpha k+\alpha j+\alpha\beta i+\alpha)\rho}\right)^{(\alpha k+\alpha j+\alpha\beta i+\alpha+1+r)}\right], & r = 1,2,3,\ldots \\ 0, & elsewhere \end{cases}$$

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From equation (4.56), it follows, therefore, that:

$$E(X) = W_{ijk}\left[(\alpha k + \alpha j + \alpha\beta i + \alpha + 2)!\left(\frac{\varphi}{(\alpha k+\alpha j+\alpha\beta i+\alpha)\rho}\right)^{(\alpha k+\alpha j+\alpha\beta i+\alpha+2)}\right] \quad (4.56a)$$

$$E(X^2) = W_{ijk}\left[(\alpha k + \alpha j + \alpha\beta i + \alpha + 3)!\left(\frac{\varphi}{(\alpha k+\alpha j+\alpha\beta i+\alpha)\rho}\right)^{(\alpha k+\alpha j+\alpha\beta i+\alpha+3)}\right] \quad (4.56b)$$

$$E(X^3) = W_{ijk}\left[(\alpha k + \alpha j + \alpha\beta i + \alpha + 4)!\left(\frac{\varphi}{(\alpha k+\alpha j+\alpha\beta i+\alpha)\rho}\right)^{(\alpha k+\alpha j+\alpha\beta i+\alpha+4)}\right] \quad (4.56c)$$

$$E(X^4) = W_{ijk}\left[(\alpha k + \alpha j + \alpha\beta i + \alpha + 5)!\left(\frac{\varphi}{(\alpha k+\alpha j+\alpha\beta i+\alpha)\rho}\right)^{(\alpha k+\alpha j+\alpha\beta i+\alpha+5)}\right] \quad (4.56d)$$

The moment generating function of EKJ distribution is given by

$$M_x(t) = \int_0^\infty e^{tx} f(x)\,dx \quad (4.57a)$$

Using the fact that $e^{tx} = \sum_{r=0}^\infty \frac{(tx)^r}{r!}$, results in:

$$M_x(t) = \int_0^\infty \sum_{r=0}^\infty \frac{(tx)^r}{r!} f(x)\,dx \quad (4.57b)$$

$$M_x(t) = \int_0^\infty \sum_{r=0}^\infty \frac{(t)^r}{r!} x^r f(x)\,dx$$

$$M_x(t) = \sum_{r=0}^\infty \frac{(t)^r}{r!} \int_0^\infty x^r f(x)\,dx$$

$$M_x(t) = \sum_{r=0}^\infty \frac{(t)^r}{r!} E(X^r) \quad (4.57c)$$

$$M_x(t) = \sum_{r=0}^\infty \frac{(t)^r}{r!} W_{ijk}\left[(\alpha k + \alpha j + \alpha\beta i + \alpha + 1 + r)!\left(\frac{\varphi}{(\alpha k+\alpha j+\alpha\beta i+\alpha)\rho}\right)^{(\alpha k+\alpha j+\alpha\beta i+\alpha+1+r)}\right]$$

$$M_x(t) = W_{ijk}\sum_{r=0}^\infty \frac{(t)^r}{r!}\left[(\alpha k + \alpha j + \alpha\beta i + \alpha + 1 + r)!\left(\frac{\varphi}{(\alpha k+\alpha j+\alpha\beta i+\alpha)\rho}\right)^{(\alpha k+\alpha j+\alpha\beta i+\alpha+1+r)}\right] \quad (4.57d)$$

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Maximum likelihood Estimation of EKJ

This section presents the maximum likelihood estimation of the proposed five-parameter Exponentiated Kumaraswamy Janardan probability distribution.

Intuitively, given parameter space $\emptyset = [\alpha \;\; \beta \;\; \rho \;\; \varphi \;\; \gamma]^T$, MLE $L(\emptyset, x)$ is given as:

$$L(\emptyset, x) = \prod_{i=1}^{n} f(x_i/\emptyset) \tag{4.58}$$

Plugging equation (4.49) into equation (4.58) results in:

$$L(\emptyset, x) = \prod_{i=1}^{n} \left[\gamma\beta\alpha\left[1-\left(1-\left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}xe^{-\frac{\rho x}{\varphi}}\right)^{\alpha}\right)^{\beta}\right]^{\gamma-1}\left[1-\left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}xe^{-\frac{\rho x}{\varphi}}\right)^{\alpha}\right]^{\beta-1}\left[-\frac{\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}e^{-\frac{\rho x}{\varphi}}\right]^{\alpha-1}\left[\frac{\rho}{\varphi}-\frac{\rho\varphi}{(\rho+\varphi^2)}+\frac{\rho x}{\varphi}\right]e^{-\frac{\rho x}{\varphi}}\right]$$

$$L(\emptyset, x) = [\gamma\beta\alpha]^n \prod_{1}^{n}\left\{\left[1-\left(1-\left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}xe^{-\frac{\rho x}{\varphi}}\right)^{\alpha}\right)^{\beta}\right]^{\gamma-1} \times \left[\left[1-\left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}xe^{-\frac{\rho x}{\varphi}}\right)^{\alpha}\right]^{\beta-1}\left[-\frac{\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}e^{-\frac{\rho x}{\varphi}}\right]^{\alpha-1}\right] \times \left[\frac{\rho}{\varphi}-\frac{\rho\varphi}{(\rho+\varphi^2)}+\frac{\rho x}{\varphi}\right]e^{-\frac{\rho}{\varphi}x}\right\} \tag{4.59}$$

Taking "ln" on both sides to obtain log-likelihood (l) as follows:

$$l = n * ln(\gamma\beta\alpha) + \sum_{i=1}^{n} ln\left\{\left[1-\left(1-\left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}xe^{-\frac{\rho x}{\varphi}}\right)^{\alpha}\right)^{\beta}\right]^{\gamma-1} \times \left[\left[1-\left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}xe^{-\frac{\rho x}{\varphi}}\right)^{\alpha}\right]^{\beta-1}\left[-\frac{\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}e^{-\frac{\rho x}{\varphi}}\right]^{\alpha-1}\right] \times \left[\frac{\rho}{\varphi}-\frac{\rho\varphi}{(\rho+\varphi^2)}+\frac{\rho x}{\varphi}\right]e^{-\frac{\rho}{\varphi}x}\right\} \tag{4.60a}$$

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$$l = n * ln(\gamma\beta\alpha) + \sum_{i=1}^{n} \left\{ ln \left[1 - \left(1 - \left(-\frac{\rho\varphi^2}{\varphi(\rho + \varphi^2)} x e^{-\frac{\rho x}{\varphi}} \right)^{\alpha} \right)^{\beta} \right]^{\gamma-1} + ln \left[\left[1 - \left(-\frac{\rho\varphi^2}{\varphi(\rho + \varphi^2)} x e^{-\frac{\rho x}{\varphi}} \right)^{\alpha} \right]^{\beta-1} + ln \left[-\frac{\rho\varphi^2 x}{\varphi(\rho + \varphi^2)} e^{-\frac{\rho x}{\varphi}} \right]^{\alpha-1} \right] + ln \left[\frac{\rho}{\varphi} - \frac{\rho\varphi}{(\rho + \varphi^2)} + \frac{\rho x}{\varphi} \right] - \frac{\rho}{\varphi} x \right\}$$

$$l = n * ln(\gamma\beta\alpha) + \sum_{i=1}^{n} \left\{ (\gamma - 1) ln \left[1 - \left(1 - \left(-\frac{\rho\varphi^2}{\varphi(\rho + \varphi^2)} x e^{-\frac{\rho x}{\varphi}} \right)^{\alpha} \right)^{\beta} \right]^{1} + \left[(\beta - 1) ln \left[1 - \left(-\frac{\rho\varphi^2}{\varphi(\rho + \varphi^2)} x e^{-\frac{\rho x}{\varphi}} \right)^{\alpha} \right]^{1} + (\alpha - 1) ln \left[-\frac{\rho\varphi^2 x}{\varphi(\rho + \varphi^2)} e^{-\frac{\rho x}{\varphi}} \right] \right] + ln \left[\frac{\rho}{\varphi} - \frac{\rho\varphi}{(\rho + \varphi^2)} + \frac{\rho x}{\varphi} \right] - \frac{\rho}{\varphi} x \right\}$$

$$l = n * ln(\gamma\beta\alpha) + \sum_{i=1}^{n} \left\{ (\gamma - 1) ln \left[1 - \left(1 - \left(-\frac{\rho\varphi^2}{\varphi(\rho + \varphi^2)} x e^{-\frac{\rho x}{\varphi}} \right)^{\alpha} \right)^{\beta} \right]^{1} + \left[(\beta - 1) ln \left[1 - \left(-\frac{\rho\varphi^2}{\varphi(\rho + \varphi^2)} x e^{-\frac{\rho x}{\varphi}} \right)^{\alpha} \right]^{1} + (\alpha - 1) ln \left[-\frac{\rho\varphi^2 x}{\varphi(\rho + \varphi^2)} e^{-\frac{\rho x}{\varphi}} \right]^{1} \right] + ln \left[\frac{\rho}{\varphi} - \frac{\rho\varphi}{(\rho + \varphi^2)} + \frac{\rho x}{\varphi} \right] - \frac{\rho}{\varphi} x \right\}$$

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$$l = n * ln(\gamma\beta\alpha)$$
$$+ \sum_{i=1}^{n} \left\{ (\gamma - 1) ln \left[1 - \left(1 - \left(-\frac{\rho\varphi^2}{\varphi(\rho + \varphi^2)} x e^{-\frac{\rho x}{\varphi}} \right)^{\alpha} \right)^{\beta} \right] \right.$$
$$+ \left[(\beta - 1) ln \left[1 - \left(-\frac{\rho\varphi^2}{\varphi(\rho + \varphi^2)} x e^{-\frac{\rho x}{\varphi}} \right)^{\alpha} \right] \right.$$
$$\left. + (\alpha - 1) \ln \left[-\frac{\rho\varphi^2 x}{\varphi(\rho + \varphi^2)} e^{-\frac{\rho x}{\varphi}} \right] \right] + ln \left[\frac{\rho}{\varphi} - \frac{\rho\varphi}{(\rho + \varphi^2)} + \frac{\rho x}{\varphi} \right] - \frac{\rho}{\varphi} x \right\}$$

$$l = n * ln(\gamma) + n * ln(\beta) + n * ln(\alpha)$$
$$+ \sum_{i=1}^{n} \left\{ (\gamma - 1) ln \left[1 - \left(1 - \left(-\frac{\rho\varphi^2}{\varphi(\rho + \varphi^2)} x e^{-\frac{\rho x}{\varphi}} \right)^{\alpha} \right)^{\beta} \right] \right.$$
$$+ \left[(\beta - 1) ln \left[1 - \left(-\frac{\rho\varphi^2}{\varphi(\rho + \varphi^2)} x e^{-\frac{\rho x}{\varphi}} \right)^{\alpha} \right] \right.$$
$$\left. + (\alpha - 1) \ln \left[-\frac{\rho\varphi^2 x}{\varphi(\rho + \varphi^2)} e^{-\frac{\rho x}{\varphi}} \right] \right] + ln \left[\frac{\rho}{\varphi} - \frac{\rho\varphi}{(\rho + \varphi^2)} + \frac{\rho x}{\varphi} \right] - \frac{\rho}{\varphi} x \right\}$$

$$l = n * ln(\gamma) + n * ln(\beta) + n * ln(\alpha)$$
$$+ \sum_{i=1}^{n} \left\{ (\gamma - 1) \ln \left[1 - \left(1 - \left(-\frac{\rho\varphi^2}{\varphi(\rho + \varphi^2)} x e^{-\frac{\rho x}{\varphi}} \right)^{\alpha} \right)^{\beta} \right] \right\}$$
$$+ + \sum_{i=1}^{n} \left\{ \left[(\beta - 1) ln \left[1 - \left(-\frac{\rho\varphi^2}{\varphi(\rho + \varphi^2)} x e^{-\frac{\rho x}{\varphi}} \right)^{\alpha} \right] \right] \right\}$$
$$+ \sum_{i=1}^{n} \left\{ \left[(\alpha - 1) \ln \left[-\frac{\rho\varphi^2 x}{\varphi(\rho + \varphi^2)} e^{-\frac{\rho x}{\varphi}} \right] \right] \right\}$$
$$+ \sum_{i=1}^{n} \left\{ ln \left[\frac{\rho}{\varphi} - \frac{\rho\varphi}{(\rho + \varphi^2)} + \frac{\rho x}{\varphi} \right] - \frac{\rho}{\varphi} x \right\}$$

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$$l = n * ln(\gamma) + n * ln(\beta) + n * ln(\alpha) + (\gamma - 1) \sum_{i=1}^{n} \ln \left[1 - \left(1 - \left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)} x e^{-\frac{\rho x}{\varphi}}\right)^{\alpha}\right)^{\beta}\right] + (\beta - 1) \sum_{i=1}^{n} ln \left[1 - \left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)} x e^{-\frac{\rho x}{\varphi}}\right)^{\alpha}\right] + \sum_{i=1}^{n} \ln \left[-\frac{\rho\varphi^2 x}{\varphi(\rho+\varphi^2)} e^{-\frac{\rho x}{\varphi}}\right] + (\alpha - 1) \sum_{i=1}^{n} ln \left[\frac{\rho}{\varphi} - \frac{\rho\varphi}{(\rho+\varphi^2)} + \frac{\rho x}{\varphi}\right] + -\frac{\rho}{\varphi} n\bar{x} \quad (4.60b)$$

The objective here is to find the values of the respective parameters in the model that maximise the likelihood function over the parameter space, $\emptyset$. This objective can be achieved if the log-likelihood function is partially differentiated over the parameter space ($\frac{\partial l}{\partial \alpha}, \frac{\partial l}{\partial \beta}\ \frac{\partial l}{\partial \rho}\ \frac{\partial l}{\partial \varphi}\ and\ \frac{\partial l}{\partial \gamma}$).

The estimated point in the parameter space $(\hat{\varphi}, \hat{\rho}, \hat{\alpha}, \hat{\beta},\ \hat{\gamma})$ that maximises the likelihood function of EKJ distribution is achieved by simultaneously solving the optimality equations:

$$\frac{\partial l}{\partial \alpha} = 0,\ \frac{\partial l}{\partial \beta} = 0,\ \frac{\partial l}{\partial \rho} = 0,\ \frac{\partial l}{\partial \varphi} = 0\ \ and\ \ \frac{\partial l}{\partial \gamma} = 0$$

The optimality equations are complex to solve analytically but can be solved empirically using an iterative technique like the Simulation annealing algorithm (a package in R-Studio).

4.3.5 Graphical Presentation of EKJ

This section presents the behaviour of Exponentiated Kumaraswamy Janardan distribution pictorially. Figure 4.12 demonstrates the behavior of EKJ for some random values of the parameters. Figure 4.13 through figure 4.17 are demonstrations of the effect of each parameter on the functional behavior of the proposed distribution.

Examination of the plots below reveals EKJ demonstrating both symmetric and asymmetric behavior depending on the parameter estimates. This behavior is an indication of the flexibility of the proposed distribution in modeling lifetime data economically.

A model that exhibits symmetric behavior offers the modeler the opportunity to subdivide the dataset to work with a symmetric portion instead of an entire set. This could save significant analytic time, analytic energy as well as analytic cost.

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The model also exhibits skewness (asymmetric) - a distortion from normality to some extent. Many lifetime data normally do not follow the assumptions of normality. Whenever assumptions of normality fail, standard deviation as a measure of dispersion provides a misleading foundation for decision-making. In this case, skewness is a highly recommended statistic in risk assessment. The skewness behavior of this proposed distribution is a pointer to the robustness of the distribution. This is the reason why this new proposed distribution is pictorially depicting the ability in modeling variability in the dataset as well as skewness in the dataset.

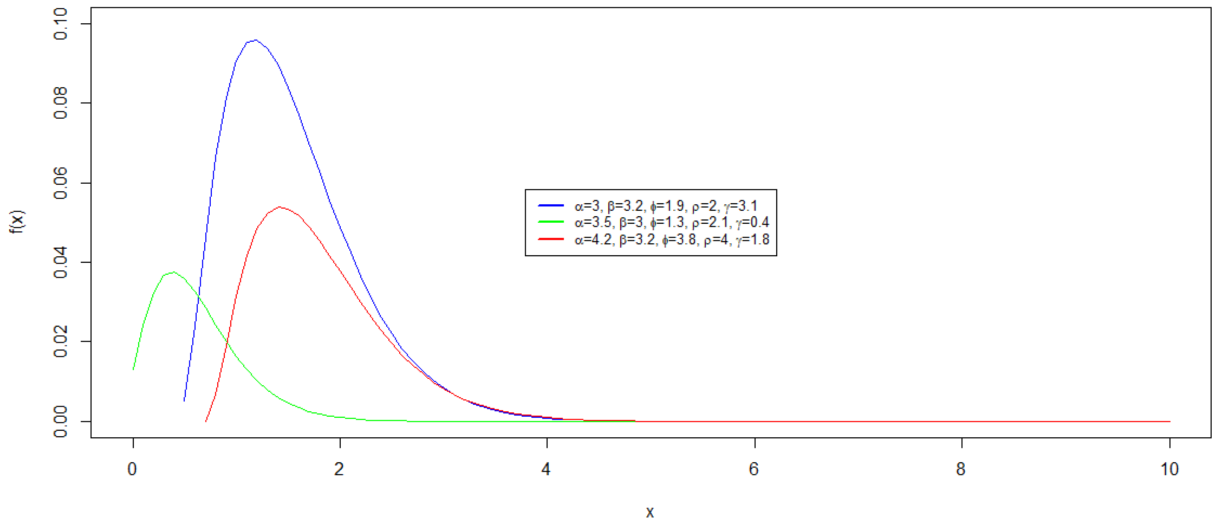

Figure 4.13 Behaviour of PDF of EKJ for some parameters

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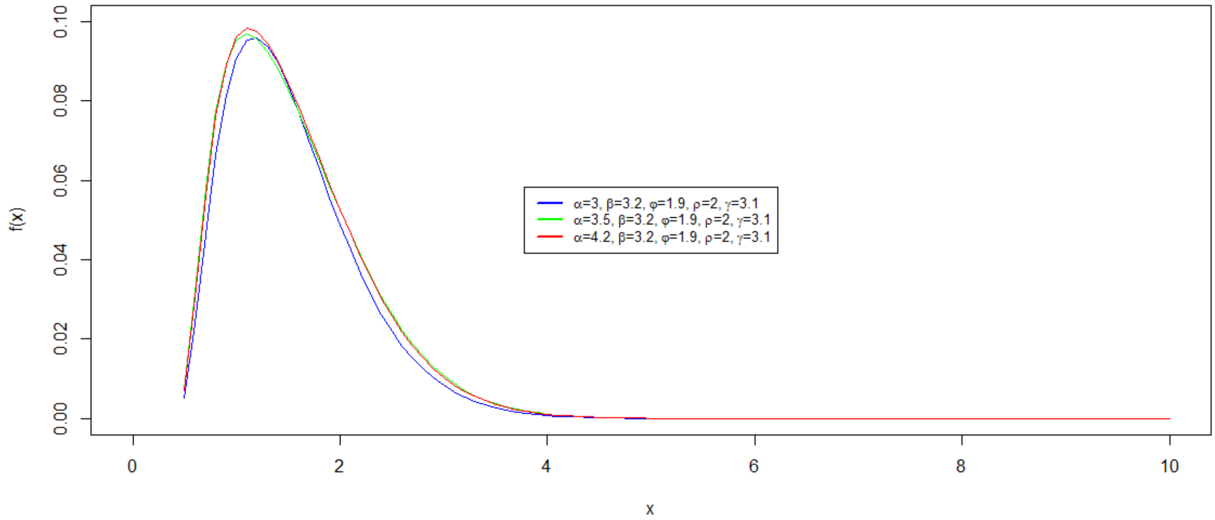

Figure 4.14 Effect of α on behaviour of PDF of EKJ

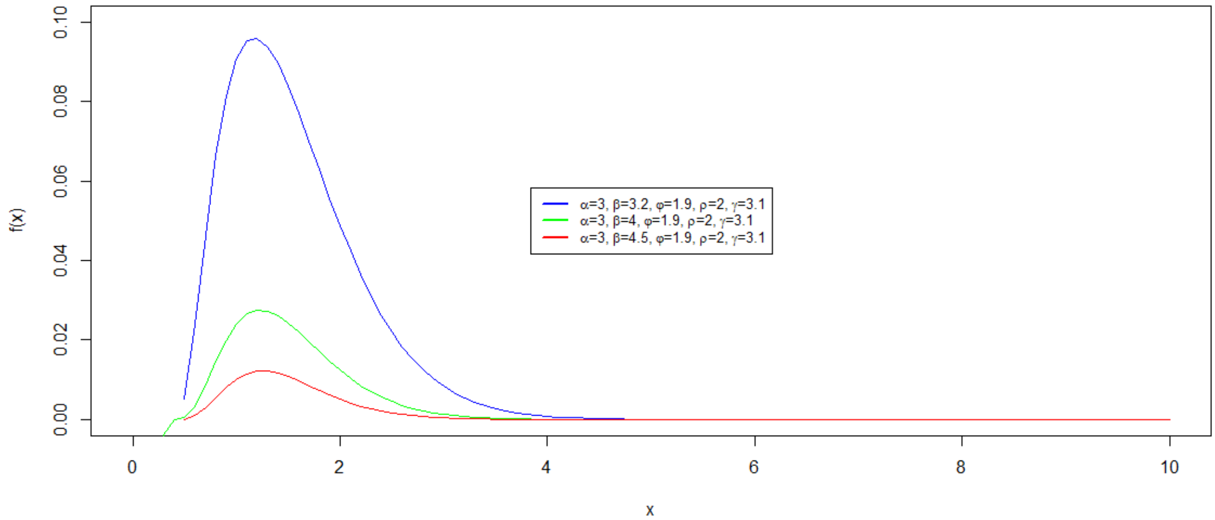

Figure 4.15 Effect of β on behaviour of PDF of EKJ

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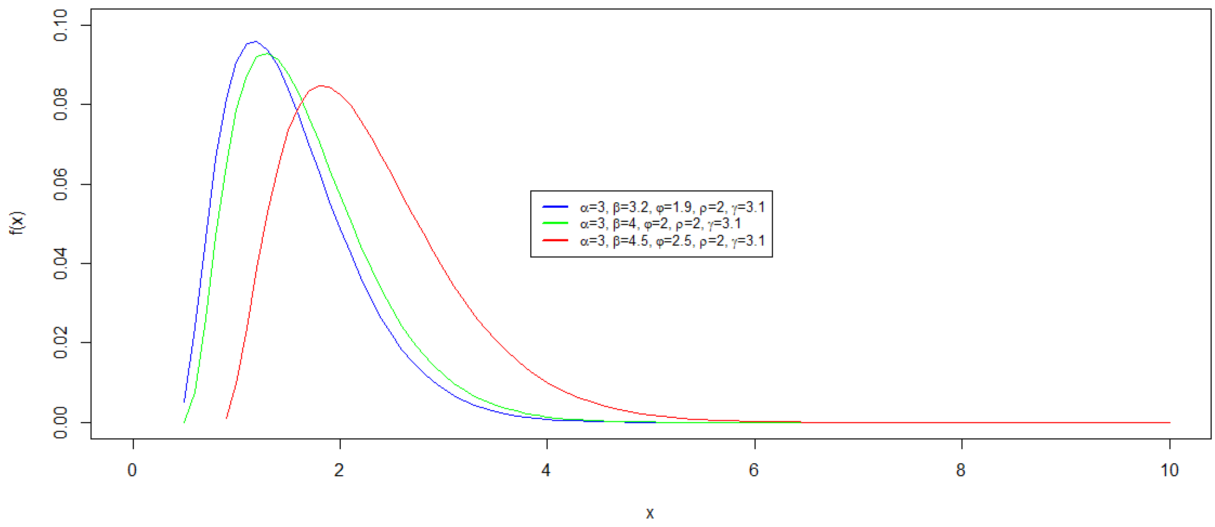

Figure 4.16 Effect of φ on behaviour of PDF of EKJ

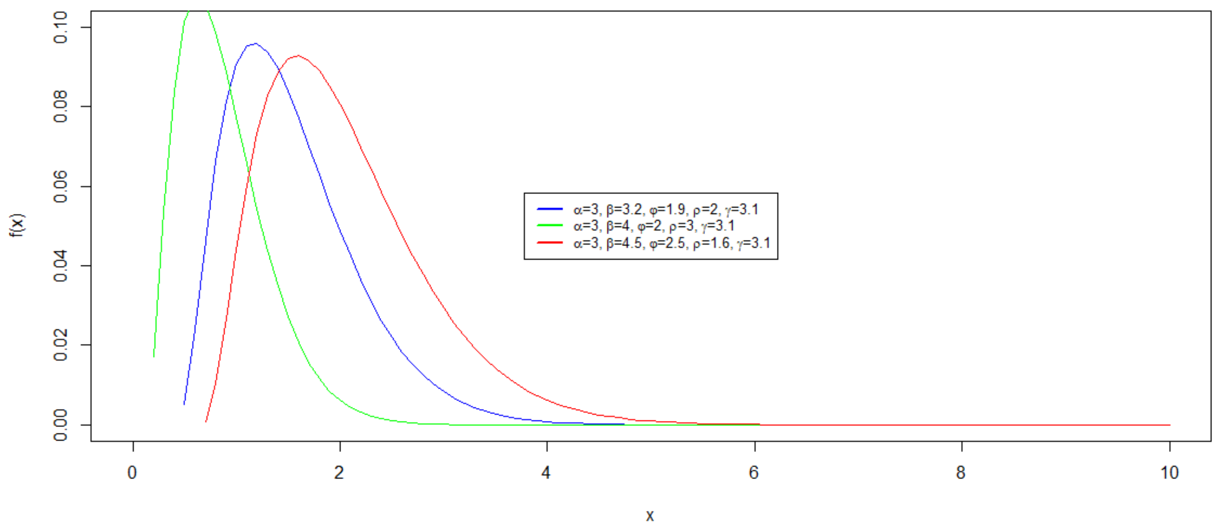

Figure 4.17 Effect of ρ on behaviour of PDF of EKJ

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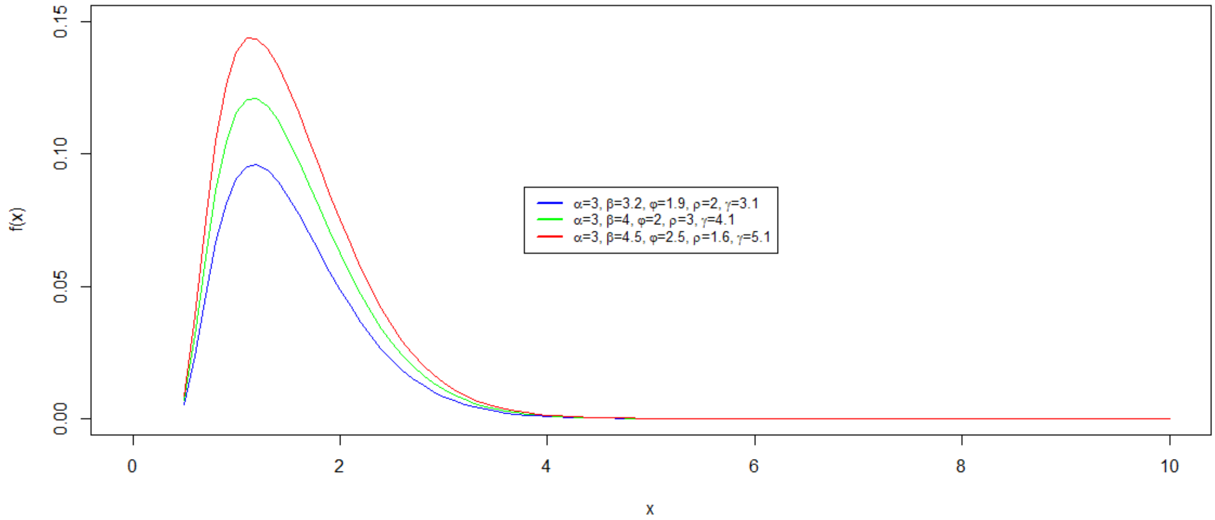

Figure 4.18 Effect of γ on behaviour of PDF of EKJ

4.3.6 Survivor and Hazard Functions of EKJ

The survival plot is a plot of the survival function. Survival function, by definition, is the probability of performing a given specified function under a given condition for a specified period. This function is also known as the reliability function (mainly, in the field of engineering). However, it is commonly known as survivor function in a wider range of applicable fields such as finance, insurance, biology, and so on.

In the field of engineering, the survival function measures the probability that a product will perform without fail for a designed lifetime under designed operative conditions.

According to Finkelstein (2008), reliability is a quality requirement of consumers from producers. The quantitative value of the survival function is a major determinant in determining the warranty of a product by suppliers/producers thereby instilling user confidence in the product.

Mathematically, the survival function of distribution is given as:

$$R(x) = 1 - F(x) \tag{4.61}$$

Substituting equation (4.47) into equation (4.61) results in:

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$$R(x) = 1 - \left[1 - \left(1 - \left[1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)} e^{-\frac{\rho}{\varphi}x}\right]^{\alpha}\right)^{\beta}\right]^{\gamma}$$

Intuitively,

$$R(x) = \int_x^{\infty} f(x)dx = 1 - \left[1 - \left(1 - \left[1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)} e^{-\frac{\rho}{\varphi}x}\right]^{\alpha}\right)^{\beta}\right]^{\gamma} \quad (4.62)$$

The close function to the survivor rate is the hazard rate. In some contexts, important characteristics of a model can be clearly demonstrated through hazard function (or failure rate) more clearly than other rates. Analytically, the hazard rate function is mathematically demonstrated as:

$$h(x) = \lim_{\Delta x \to 0} \frac{prob(x<X<x+\Delta x I X>x)}{\Delta x} \quad (4.63)$$

Using the Bayes theorem of conditional probability;

$$h(x) = \lim_{\Delta x \to 0} \frac{F(x+\Delta x)-F(x)}{\Delta x\,(1-F(x))} \quad (4.64)$$

$$h(x) = \frac{f(x)}{1-F(x)} = \frac{f(x)}{R(x)} \quad (4.65)$$

Hazard rate plays a very central role in survival evaluation. It is mainly used in the fields of demography, actuary, and epidemiological studies. It is commonly known as the force of mortality. Hazard rate is also applicable in risk evaluation in finance. In engineering, the hazard rate of a machine or device with life "X" conforming to a distribution, is the instantaneous conditional probability of failure given that the machine/device has served until time "x".

The quantitative value of hazard rate $h(x)$ has myriad uses in describing many life phenomena. Keenly, it has numerous applications in myriad fields including (but not limited to) engineering, economics, insurance, epidemiology, and so on.

More closely, the hazard function of Exponentiated Kumaraswamy Janardan distribution is derived as:

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$$h(x) = \frac{W_{ijk}\left[x^{\alpha(k+j+\beta i+1)} \times e^{-\alpha(k+j+\beta i+1)\frac{\rho x}{\varphi}}\right]}{1-\left[1-\left(1-\left[1-\frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\, e^{-\frac{\rho}{\varphi}x}\right]^{\alpha}\right)^{\beta}\right]^{\gamma}} \tag{4.66}$$

In expanded form, we obtain:

$$h(x) = \left\{\left[\gamma\beta\alpha\left[1-\left(1-\left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}xe^{-\frac{\rho x}{\varphi}}\right)^{\alpha}\right)^{\beta}\right]^{\gamma-1}\left[1 - \left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}xe^{-\frac{\rho x}{\varphi}}\right)^{\alpha}\right]^{\beta-1}\left[-\frac{\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}e^{-\frac{\rho x}{\varphi}}\right]^{\alpha-1}\left[\frac{\rho}{\varphi}-\frac{\rho\varphi}{(\rho+\varphi^2)} + \frac{\rho x}{\varphi}\right]e^{-\frac{\rho x}{\varphi}}\right]\right\} \div$$

$$\left\{1-\left[1-\left(1-\left[1-\frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\, e^{-\frac{\rho}{\varphi}x}\right]^{\alpha}\right)^{\beta}\right]^{\gamma}\right\}$$

$$h(x) = \frac{\left[\gamma\beta\alpha\left[1-\left(1-\left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}xe^{-\frac{\rho x}{\varphi}}\right)^{\alpha}\right)^{\beta}\right]^{\gamma-1}\left[1-\left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}xe^{-\frac{\rho x}{\varphi}}\right)^{\alpha}\right]^{\beta-1}\times \left[-\frac{\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}e^{-\frac{\rho x}{\varphi}}\right]^{\alpha-1}\left[\frac{\rho}{\varphi}-\frac{\rho\varphi}{(\rho+\varphi^2)}+\frac{\rho x}{\varphi}\right]e^{-\frac{\rho x}{\varphi}}\right]}{1-\left[1-\left(1-\left[1-\frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\, e^{-\frac{\rho}{\varphi}x}\right]^{\alpha}\right)^{\beta}\right]^{\gamma}} \tag{4.67}$$

The hazard rate function can exhibit a rate that is constant, increasing, decreasing, or upside-down bathtub and bathtub depending on the values of parameters of the model. Any model that exhibits all the above shapes has a great ability in robustness in modeling hazard rates appropriately. Succinctly, a distribution with a robust hazard rate can model data that exhibit constant failure rate, monotonically increasing (decreasing) failure rate, and bathtub failure rate.

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CHAPTER 5

EMPIRICAL RESULTS

5.0 Introduction

This section presents the empirical results of the study. Empirical findings are presented in three subsections according to the models proposed and developed. The three models developed are fed with five different datasets.

5.1 Application of Exponentiated Janardan to Lifetime Data

This subsection presents the results of applying Exponentiated Janardan distribution as well as Janardan distribution to lifetime data. Dataset 1 was one of the datasets Shanker (2013) used to test the Janardan distribution and found it to be better than the Lindley distribution. Datasets 2 and 4 are obtained from MTN Ghana Call Center. While dataset 2 was obtained from the Low Value (LV) queue, dataset 4 is obtained from the High Value (HV) queue. Datasets 3 and 5 are the academic score of students from Bluecrest University (a private university) and Accra Technical University (a public university) respectively. Both Janardan and Exponentiated Janardan distributions were fitted to the five different datasets concurrently. In all cases, Exponentiated Janardan distribution provides a smaller AIC than Janardan distribution.

Table 5.1 The empirical optimisation result of Exponentiated Janardan Distribution

Datasets	Model	Mean	SD	$\hat{\varphi}$	$\hat{\rho}$	$\hat{\gamma}$	-(log L)	AIC
Dataset 1	EJ	0.73	0.71	0.295	0.598	1.411	15.9735	37.947
	J			2.419	5.044	-----	17.6080	39.216
Dataset 2 (Queue LV)	EJ	1.49	0.35	$3.7*10^{-8}$	$8.5*10^{-5}$	0.104	-702.77	-696.77
	J			46.43	17.69	------	-91.32	-87.32
Dataset 3 (examscoreBC)	EJ	0.87	0.25	0.13	0.09	1.25	30.12	36.12
	J			2.78	2,01	-----	33.768	37.768
Dataset 4 (Queue HV)	EJ	2.49	0.5	0.007	0.089	0.111	-216.76	-201.76

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	J			1.437	1.96	------	-112,51	- 108.51
Dataset 5 (examscoreatu)	EJ	0.76	0.15	0.17	0.11	1.25	31.17	37.17
	J			2.38	2.17	-----	33.89	37.89

5.2 Application of Kumaraswamy Janardan to Lifetime Data

This subsection presents the empirical optimisation result of the proposed Kumaraswamy-Janardan distribution using real data to demonstrate that the K-J model provides a significant improvement over the sub-models (Lindley and Janardan). Datasets 1, 2, and 3 are obtained from the Janardan article. These datasets were fitted to Lindley, Janardan and Kumaraswamy-Janardan distributions concurrently. The optimisation output is presented in table 5.2.

Table 5.2 Empirical optimisation result of Kumaraswamy-Janardan Model

Datasets	Model	$\hat{\rho}$	$\hat{\varphi}$	$\hat{\alpha}$	$\hat{\beta}$	-(log L)	AIC
Dataset 1	Lindley	2.9097	-	-	-	73.1047	75.1047
	Janardan	64.3228	4.6215	-	-	- 125.7309	- 121.7309
	K-J	4.3964	0.1174	13.0521	0.9394	- 231.1150	- 223.1150
Dataset 2	Lindley	2.9097	-	-	-	-28.8451	-26.8451
	Janardan	76.4254	4.9416	-	-	-33.5370	-29.5370
	K-J	0.1908	0.2571	7.0334	15.7885	-39.7005	-31.7005
Dataset 3	Lindley	2.7243	-	-	-	95.9087	97.9087
	Janardan	71.0587	4.8012	-	-	37.0897	78.1794
	K-J	2.7607	10.8387	5.5524	9.6799	29.5206	67.0522

From table 5.2, the Maximum Likelihood Estimate for each parameter of the given model is presented alongside the loglikelihood value which eventually feeds into AIC. The MLE (in table 5.2) are estimated point(s) in the parameter space that maximizes the likelihood function of each of the probability models in the table with the given datasets. These

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estimates are helping to make inferences about the population that is most likely to have generated the sample datasets. To determine which model among the considered models is the best in fitting these sample data, the researcher considered the Akaike Information Criterion (AIC). The model with the least AIC is the best among the considered models. From the table, it is observed that Kumaraswamy-Janardan (K-J) records the lowest AIC and hence is the best of the considered models.

5.3 Application of Exponentiated Kumaraswamy Janardan to Lifetime Data

This subsection presents the empirical result of the optimisation of Exponentiated Kumaraswamy Janardan (EKJ) distribution. This distribution is run alongside the sub-models on five (5) different datasets and suitability is determined. Datasets 1,2,3,4 and 5 were retrieved from the literature. These datasets were previously fitted to Lindley and Janardan distributions. The optimality output with the help of simulated annealing is presented in table 5.3.

Table 5.3 Empirical optimisation result of Exponentiated Kumaraswamy Janardan Distribution

Data sets	Model	Mean	SD	$\hat{\rho}$	$\hat{\varphi}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	-(log L)	AIC
1	L	2.52	0.25	2.910	-	-	-	-	73.105	75.105
	J			64.32	4.62	-	-	-	-125.731	-121.73
	KJ			4.396	0.117	13.052	0.939	-	-231.115	-223.12
	EJ			0.32	1.07	---	---	0.15	-229.31	-223.31
	EKJ			5.98	0.982	2.992	0.006	0.59	-232.25	-222.25
2	L	172	30	2.910	-	-	-	-	-28.8451	-26.8451
	J			76.425	4.942	-	-	-	-33.537	-29.5370
	KJ			0.191	0.257	7.033	15.789	-	-39.7005	-31.7005
	EJ			0.378	3.001	---	---	0.021	-36.28	-30.28
	EKJ			3.215	0.891	5.340	4.952	0.369	-39.789	-29.789
3	L			2.724	-	-	-	-	95.9087	97.9087
	J	0.98	0.08	71.059	4.801	-	-	-	37.9897	41.9897
	KJ			2.761	10.839	5.552	9.68	-	29.5206	37.5206
	EJ			2.00	5.25	-	-	0.978	31.950	37.95

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	EKJ			7.23	0.09	8.21	5.20	1.09	31.595	41.595
4	L	21.3	5.7	2.731	-	-	-	-	27.326	29.326
	J			5.044	2.419	-	-	-	17.6080	21.608
	KJ			6.982	5.001	0.965	5.941	-	11.289	19.289
	EJ			5.598	2.295	-	-	1.411	15.9735	21.9735
	EKJ			0.98	2.038	8.321	4.231	0.992	14.852	24.852
5	L	37.2	6	11.281	-	-	-	-	231.58	233.58
	J			5.28	6.245	-	-	-	227.620	231.620
	KJ			2.212	4.357	1.372	0.978	-	198.83	206.83
	EJ			3.891	0.924	-	-	1.285	200.94	206.94
	EKJ			5.126	1.732	5.203	0.872	0.901	199.72	209.72

Table 5.3 presents the Maximum Likelihood Estimate (MLE) for each parameter of the given model as well as the loglikelihood value which eventually feeds into AIC. The MLE is the estimated point(s) in the parameter space that maximises the likelihood function of each of the probability models in the table with the given datasets. These estimates are helping to make inferences about the population that is most likely to have generated the sample datasets. The essence of the estimates in probability theory is to obtain a suitable model for better projection. The values of the estimates culminate in calculating the likelihood value of a given model. The models considered are Lindley, Janardan, Kumaraswamy-Janardan (new), Exponentiated Janardan (new), and Exponentiated Kumaraswamy Janardan (new). The three new models are fitted to five different datasets. These datasets were used in judging the superiority of Janardan over Lindley. In order to determine which model among the considered models is the best in fitting these sample data, the researcher, considered the Akaike Information Criterion (AIC). The model with the least AIC is the best among the considered models.

From table 5.3, it can be observed that all the new models (Exponentiated Janardan, Kumaraswamy Janardan and Exponentiated Kumaraswamy Janardan) independently recorded the lowest AIC value in comparison to the existing models (Lindley and Janardan). By implication, the new models provide a better fit to all the considered sample datasets than the existing sub-models.

Recalling that the three new models developed in this project are a three-parameter Exponentiated Janardan Distribution, a four-parameter Kumaraswamy Janardan

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Distribution, and a five-parameter Exponentiated Janardan Distribution. Apart from the fact that these three distributions show superiority over Janardan Distribution and its sub-model (Lindley distribution), the study further investigates the goodness of fit among the three new models. In comparing the three new distributions, Kumaraswamy Janardan (KJ) Distribution proves superior in most cases. KJ recorded the lowest AIC in all but one dataset. In dataset 1, the AIC for Exponentiated Janardan (EJ) is -223.31 while KJ is -223.115 and EKJ is -222.25. In this dataset, EJ proves to be the best among these new models even though the KJ's AIC value is very close. However, the difference in these AICs is less than 2 (in absolute terms), indicating that they are statistically the same with the dataset. In the case of datasets 2, 3, 4 and 5; KJ recorded the lowest AIC which was consistently followed by EJ.

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CHAPTER 6

DISCUSSION, CONCLUSIONS AND RECOMMENDATIONS

6.1 Discussions

This study came out with three new probability distributions of which the Janardan distribution is a particular case. The new distributions are run alongside their sub-models in the data environment. These new distributions have exponential distribution as a classical baseline model. Since exponential distribution is the classical baseline, the new distributions can be used to model zero bounded variables in the field of industrial quality control as suggested by Epstein (1958). It can also be used in modeling the stochastic theory of accident and survival as suggested by Esary (1957). These new distributions have the potential in modeling variables in reliability engineering and survival analysis since their hazard rate exhibit bathtub characteristics.

As indicated in chapter two the important functions in stochastic modeling in the area of survival and reliability analysis are survivor function and hazard function. The drawbacks of the constant hazard function (in the case of exponential distribution) and the monotonic hazard rate function (in the case of Weibull and Janardan distributions) are corrected in these new distributions. Each of the new distributions exhibits a constant rate, monotonically increasing/decreasing rate as well as bathtub rate (upside down and downside up). These characteristics of the hazard function of the new distributions are a kind of superiority of the new distributions over the sub-models (Exponential, Weibull, Lindley and Janardan distributions) as well as competing models (Size-Biased Janardan, Transmuted Janardan distribution, etc)

Although Bashir and Rasul (2016) claimed that the Janardan distribution is one of the important distributions for the life model and it has many applications in real-life data, the empirical result of this study reveals that all three new distributions provide a better fit than Janardan.

6.2 Conclusions

The ideas of probability have evolved over the years from the game of chance to many practical and scientific problems such as in the areas of theory of errors, actuarial

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mathematics, statistical mechanics, etc. Due to the evolution of probabilistic ideas, more probability theories/distributions are being developed from time to time to improve the ability of existing distributions to model. One of the new improved distributions that attracted the attention of the researcher is the Janardan distribution. This distribution had been tried and tested on many lifetime datasets and proven to be a more relaxed and better fit than Lindley and Exponential distributions.

However, Janardan distribution is discovered to be limited in controlling skewness and kurtosis which most lifetime data exhibit, hence the need to modify the Janardan distribution through the method of parametrisation. To improve the usability and flexibility of the Janardan probability distribution, the study was designed to come out with three new probability distributions of which the Janardan distribution is the baseline, the statistical properties of these new distributions were established as well as their goodness of fit through the use of data are tested.

In line with the study objectives, three new distributions are developed through the method of parametrisation. These new distributions are Exponentiated Janardan (Three parameter distribution), Kumaraswamy Janardan (Four Parameter distribution) and Exponentiated Kumaraswamy Janardan (Five parameter distribution). Statistical properties such as PDF, CDF, Hazard rate, Survivor rate, Moments, Moment Generating function and MLE are established for each of the derived distributions.

Empirical results reveal that all the derived models provide better fit to all the considered sample datasets than the existing sub-models. Apart from the fact that these three derived distributions show superiority over Janardan Distribution and its sub-model (Lindley distribution), the study further investigates the goodness of fit among the three new models. In comparing the three new distributions, the four-parameter Kumaraswamy Janardan (KJ) Distribution proves superiority in most cases.

6.3 Recommendations

The knowledge and applications of probability distribution need to be continuously enhanced and become more flexible in modeling current lifetime data effectively and efficiently. It is against this background that the Janardan probability distribution is improved upon by the researcher. This study would bridge the knowledge gap that exists in

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probability distributions. Kumaraswamy Janardan's distribution proves its robustness with the sample data used in the study.

The researcher would like to recommend that scholars expand the statistical properties of the new distributions established in this study to bridge the research gap in mathematical computations.

Probability curriculum developers should include these new distributions in their curriculum to enhance their knowledge and usability of them since these distributions have proven to be better in modeling than the traditional classical distributions such as Exponential and Weibull distributions.

Industry experts in the fields of reliability engineering, demography, actuary, etc; should use Kumaraswamy Janardan in modeling and predicting the reliability and hazard rate of their product since this distribution provides a robust hazard rate function.

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APPENDIX

R-STUDIO CODES

Useful codes for EJ

```
### EJ PDF PLOT ##

EJ_PDF_PLOT<-function(x,f,r,g)
 {
 A=f*(r+f^2)
 B=exp(-(r/f)*x)
 C=f*(r+f^2)
 J_CDF=1-(((A+(r*(f^2)*x))/A)*B)
 J_PDF=((f^2)/C)*(1+(r*x))*B
 #  EXPONENTIATED JARNADAN CDF ###
 EJ_CDF<-(J_CDF)^g
 ######  EXPONENTIATED JARNADAN PDF #########
 EJ_PDF<-g*(J_PDF*(J_CDF)^(g-1))
        return(EJ_PDF)
}
windows(width=20,height=10)
par(mfrow=c(1,1))
curve(EJ_PDF_PLOT(x,0.1,0.2,5),0,10,col="blue",ylab=expression(paste('f',"(x)")),ylim=
c(0,0.5),xlab="x",lty=1,lwd=1)
curve(EJ_PDF_PLOT(x,0.2,0.5,3),0,10,col="green",add=TRUE,lty=1,lwd=1)
curve(EJ_PDF_PLOT(x,0.3,0.6,2),0,10,col="red",add=TRUE,lty=1,lwd=1)
legend("topright",inset=c(0.05),cex=1.0,legend=c(expression(paste(varphi,"=",0.1,",",~rho
,"=",0.2,",",~gamma,"=",5)),expression(paste(varphi,"=",0.2,",",~rho,"=",0.5,",",~gamma,
"=",0.2)),
expression(paste(varphi,"=",0.3,",",~rho,"=",0.6,",",~gamma,"=",2))),lty=1,lwd=2,col=c("
blue","green","red"))
```


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```
title(main="Figure 4.1: Behaviour of PDF of EJ

    for some parameters")

################ EJ SURVIVAL PLOT ##############################

EJ_SURVIVAL_PLOT<-function(x,f,r,g)

{


  EJ_SURVIVAL<-1-EJ_CDF

##  EJ_PDF<-g*(J_PDF*(J_CDF)^(g-1))###


  return(EJ_SURVIVAL)

}

windows(width=20,height=10)

par(mfrow=c(1,1))

curve(EJ_SURVIVAL_PLOT(x,0.1,0.2,5),0,10,col="blue",ylab=expression(paste('f',"(x)")
),ylim=c(0,1),xlab="x",lty=1,lwd=1)

curve(EJ_SURVIVAL_PLOT(x,0.2,0.5,3),0,10,col="green",add=TRUE,lty=1,lwd=1)

curve(EJ_SURVIVAL_PLOT(x,0.3,0.6,2),0,10,col="red",add=TRUE,lty=1,lwd=1)

legend("topright",inset=c(0.05),cex=1.0,legend=c(expression(paste(varphi,"=",0.1,",",~rho
,"=",0.2,",",~gamma,"=",5)),expression(paste(varphi,"=",0.2,",",~rho,"=",0.5,",",~gamma,
"=",0.2)),
expression(paste(varphi,"=",0.3,",",~rho,"=",0.6,",",~gamma,"=",2))),lty=1,lwd=2,col=c("
blue","green","red"))

title(main="Figure 4.2: Behaviour of Survival Function of EJ

    for some parameters")


##### EJ HAZARD PLOT ###

EJ_HAZARD_PLOT<-function(x,f,r,g)

{

  EJ_HAZARD<-EJ_PDF/EJ_SURVIVAL
```


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```
 return(EJ_HAZARD)

}

windows(width=20,height=10)

par(mfrow=c(1,1))

curve(EJ_HAZARD_PLOT(x,0.1,0.2,5),0,10,col="blue",ylab=expression(paste('f',"(x)")),
ylim=c(0,1),xlab="x",lty=1,lwd=1)

curve(EJ_HAZARD_PLOT(x,0.2,0.5,3),0,10,col="green",add=TRUE,lty=1,lwd=1)

curve(EJ_HAZARD_PLOT(x,0.3,0.6,2),0,10,col="red",add=TRUE,lty=1,lwd=1)

legend("topright",inset=c(0.05),cex=1.0,legend=c(expression(paste(varphi,"=",0.1,",",~rho
,"=",0.2,",",~gamma,"=",5)),expression(paste(varphi,"=",0.2,",",~rho,"=",0.5,",",~gamma,
"=",0.2)),
expression(paste(varphi,"=",0.3,",",~rho,"=",0.6,",",~gamma,"=",2))),lty=1,lwd=2,col=c("
blue","green","red"))

title(main="Figure 4.3: Behaviour of Hazard Rate of EJ

   for some parameters")



 ######################

 janardan.lik<-function(theta,x)

 {

  theta<-c(r,f)

  r<-theta[1]

 f<-theta[2]

 n<-nrow(x)

logl<- 2*n*ln(r)-n*ln(f)-n*ln(r+f^2)+n*ln(1+f*x)-(r/f)*sum(x)

 x<-
c(1.31209,0.61040,0.50733,2.17203,1.78809,0.60867,2.64002,0.57412,0.45361,0.42831,0
.34411,0.23693,0.07457,0.18864,0.38920,0.18449,0.76830,0.29148,0.67360,2.06885,0.51
296,0.16361,0.13590,0.56586

)

 return(-logl)
```


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```
  }
  optim(c(1,1),janardan.lik,x,method="BFGS")
summary(-logl)
print
######## Demostration of effect of the parameters######
EJ_PDF_PLOT<-function(x,f,r,g)
{
  ######   JARNADAN CDF  ###############
  A=f*(r+f^2)
  B=exp(-(r/f)*x)
  J_CDF=1-(((A+(r*(f^2)*x))/A)*B)
  ######  JARNADAN PDF  ##############
  C=f*(r+f^2)
  B=exp(-(r/f)*x)
  J_PDF=((f^2)/C)*(1+(r*x))*B
  #### EXPONENTITED CDF######
  ### F(x)= (G(x))^g ####
  ######  EXPONENTIATED JARNADAN CDF #########
  EJ_CDF<-(J_CDF)^g
  ##### EXPONENTIATED PDF#####
  ## f(x)= g*g(x)*(G(x))^(g-1)###
  ######  EXPONENTIATED JARNADAN PDF #########
  EJ_PDF<-g*(J_PDF*(J_CDF)^(g-1))
  return(EJ_PDF)   }
windows(width=20,height=10)
par(mfrow=c(1,1))
```


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```
curve(EJ_PDF_PLOT(x,0.1,0.2,5),0,10,col="blue",ylab=expression(paste('f',"(x)")),ylim=
c(0,0.5),xlab="x",lty=1,lwd=1)

curve(EJ_PDF_PLOT(x,0.1,0.2,10),0,10,col="green",add=TRUE,lty=1,lwd=1)

curve(EJ_PDF_PLOT(x,0.1,0.2,15),0,10,col="red",add=TRUE,lty=1,lwd=1)

legend("topleft",inset=c(0.05),cex=1.0,legend=c(expression(paste(varphi,"=",0.1,",",~rho,
"=",0.2,",",~gamma,"=",5)),expression(paste(varphi,"=",0.1,",",~rho,"=",0.2,",",~gamma,"
=",10)),
expression(paste(varphi,"=",0.1,",",~rho,"=",0.2,",",~gamma,"=",15))),lty=1,lwd=2,col=c(
"blue","green","red"))

title(main="Figure 4.4: Demostration of Effect of gamma

    on Behaviour of  PDF of EJ ")

#### Effect of Rho ####

windows(width=20,height=10)

par(mfrow=c(1,1))

curve(EJ_PDF_PLOT(x,0.1,0.2,5),0,10,col="blue",ylab=expression(paste('f',"(x)")),ylim=
c(0,0.5),xlab="x",lty=1,lwd=1)

curve(EJ_PDF_PLOT(x,0.1,0.1,5),0,10,col="green",add=TRUE,lty=1,lwd=1)

curve(EJ_PDF_PLOT(x,0.1,0.08,5),0,10,col="red",add=TRUE,lty=1,lwd=1)

legend("topright",inset=c(0.05),cex=1.0,legend=c(expression(paste(varphi,"=",0.1,",",~rho
,"=",0.2,",",~gamma,"=",5)),expression(paste(varphi,"=",0.1,",",~rho,"=",0.1,",",~gamma,
"=",5)),
expression(paste(varphi,"=",0.1,",",~rho,"=",0.08,",",~gamma,"=",5))),lty=1,lwd=2,col=c(
"blue","green","red"))

title(main="Figure 4.5: Demostration of Effect of rho

    on Behaviour of  PDF of EJ "

#### Effect of Varphi ####

windows(width=20,height=10)

par(mfrow=c(1,1))

curve(EJ_PDF_PLOT(x,0.1,0.1,5),0,10,col="blue",ylab=expression(paste('f',"(x)")),ylim=
c(0,0.5),xlab="x",lty=1,lwd=1)

curve(EJ_PDF_PLOT(x,0.5,0.1,5),0,10,col="green",add=TRUE,lty=1,lwd=1)
```


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```
curve(EJ_PDF_PLOT(x,0.6,0.1,5),0,10,col="red",add=TRUE,lty=1,lwd=1)

legend("topright",inset=c(0.05),cex=1.0,legend=c(expression(paste(varphi,"=",0.1,",",~rho
,"=",0.1,",",~gamma,"=",5)),expression(paste(varphi,"=",0.5,",",~rho,"=",0.1,",",~gamma,
"=",5)),
expression(paste(varphi,"=",0.6,",",~rho,"=",0.1,",",~gamma,"=",5))),lty=1,lwd=2,col=c("
blue","green","red"))

title(main="Figure 4.6: Demostration of Effect of Varphi

   on Behaviour of  PDF of EJ "

############################ EJ MLE
OPTIMIZATION################################

janardan_re.fit<-function(params, x_i){

 varphi<-params[1]

 rho<-params[2]

 gamma<-params[3]

 n<-length(x_i)

 x_i<-
c(1.31209,0.61040,0.50733,2.17203,1.78809,0.60867,2.64002,0.57412,0.45361,0.42831,


0.34411,0.23693,0.07457,0.18864,0.38920,0.18449,0.76830,0.29148,0.67360,2.06885,0.5
1296,0.16361,

     0.13590,0.56586)

 x_i

 logl<-n*log(gamma*rho**2/(varphi*(rho + varphi**2))) +

  sum(log((1 -((rho*varphi**2)*x_i + varphi*(rho + varphi**2))*exp(-
rho*x_i/varphi)/(varphi*(rho + varphi**2)))**(gamma - 1)*(varphi*x_i + 1)*exp(-
rho*x_i/varphi)))

  return(-logl)   }

janardan_re.est<-function(x_i){

 data.params<-optim(par = runif(3), janardan_re.fit, method = "SANN",

           x_i = data, hessian = T)

 estimates<-data.params
```


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```
  return(estimates)
}
janardan_re.est(x_i)
############################ Jan Only MLE FINAL######
jan_re.fit<-function(params, x_i){
  varphi<-params[1]
  rho<-params[2]
  n<-length(x_i)
  x_i<-
c(1.31209,0.61040,0.50733,2.17203,1.78809,0.60867,2.64002,0.57412,0.45361,0.42831,
0.34411,0.23693,0.07457,0.18864,0.38920,0.18449,0.76830,0.29148,0.67360,2.06885,0.5
1296,0.16361, 0.13590,0.56586)
  logl<-n*log(1*rho**2/(varphi*(rho + varphi**2))) +
    sum(log((1 -((rho*varphi**2)*x_i + varphi*(rho + varphi**2))*exp(-
rho*x_i/varphi)/(varphi*(rho + varphi**2)))**(1 - 1)*(varphi*x_i + 1)*exp(-
rho*x_i/varphi)))
  return(-logl)
}
jan_re.est<-function(x_i){
  data.params<-optim(par = runif(2), jan_re.fit, method = "SANN",
                     x_i = data, hessian = T)
  estimates<-data.params
  return(estimates)
}
jan_re.est(x_i)
###### EJ MLE data 2 #####
janardan_re.fit<-function(params, x_i){
  varphi<-params[1]
  rho<-params[2]
```


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```
  gamma<-params[3]

  n<-length(x_i)

  x_i<-c(0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2.00, 0.74,
1.04,

        1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76,  1.82, 2.01, 0.77, 1.11, 1.28, 1.42,

        1.50, 0.54, 1.60, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.50, 1.55,

1.61,1.62,1.66,1.70,1.77,1.84,0.84,1.24,1.30,1.48,1.51,1.55,1.61,1.63,1.67,1.70,1.78,1.89)

  x_i


  logl<-n*log(gamma*rho**2/(varphi*(rho + varphi**2))) +

    sum(log((1 -((rho*varphi**2)*x_i + varphi*(rho + varphi**2))*exp(-
rho*x_i/varphi)/(varphi*(rho + varphi**2)))**(gamma - 1)*(varphi*x_i + 1)*exp(-
rho*x_i/varphi)))

  return(-logl)

}

janardan_re.est<-function(x_i){


  data.params<-optim(par = runif(3), janardan_re.fit, method = "SANN",

                     x_i = data, hessian = T)

  estimates<-data.params

  return(estimates)

}

janardan_re.est(x_i)

#### J MLE Only ###

jan_re.fit<-function(params, x_i){

  varphi<-params[1]

  rho<-params[2]
```


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```
  n<-length(x_i)
  x_i<-c(0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2.00, 0.74,
1.04,
         1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76,  1.82, 2.01, 0.77, 1.11, 1.28, 1.42,
         1.50, 0.54, 1.60, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.50, 1.55,

1.61,1.62,1.66,1.70,1.77,1.84,0.84,1.24,1.30,1.48,1.51,1.55,1.61,1.63,1.67,1.70,1.78,1.89)
  x_i
  logl<-n*log(1*rho**2/(varphi*(rho + varphi**2))) +
    sum(log((1 -((rho*varphi**2)*x_i + varphi*(rho + varphi**2))*exp(-
rho*x_i/varphi)/(varphi*(rho + varphi**2)))**(1 - 1)*(varphi*x_i + 1)*exp(-
rho*x_i/varphi)))
  return(-logl)
}
jan_re.est<-function(x_i){
  data.params<-optim(par = runif(2), jan_re.fit, method = "SANN",
                     x_i = data, hessian = T)
  estimates<-data.params
  return(estimates)
}
jan_re.est(x_i)
#####
jan_re.fit<-function(params, x_i){
  varphi<-params[1]
  rho<-params[2]
  n<-length(x_i)
  x_i<-c(0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2.00, 0.74,
1.04,
         1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76,  1.82, 2.01, 0.77, 1.11, 1.28, 1.42,
```


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```
    1.50, 0.54, 1.60, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.50, 1.55,
1.61,1.62,1.66,1.70,1.77,1.84,0.84,1.24,1.30,1.48,1.51,1.55,1.61,1.63,1.67,1.70,1.78,1.89)
  x_i
  logl<-n*log(1*rho**2/(varphi*(rho + varphi**2))) +
    sum(log((1 -((rho*varphi**2)*x_i + varphi*(rho + varphi**2))*exp(-
rho*x_i/varphi)/(varphi*(rho + varphi**2)))**(1 - 1)*(varphi*x_i + 1)*exp(-
rho*x_i/varphi)))
  return(-logl)
}
janardan_re.est<-function(x_i){
  data.params<-optim(par = runif(2), jan_re.fit, method = "SANN",
                     x_i = data, hessian = T)
  estimates<-data.params
  return(estimates)      }
janardan_re.est(x_i)
###### EJ MLE data 3 #####
janardan_re.fit<-function(params, x_i){
  varphi<-params[1]
  rho<-params[2]
  gamma<-params[3]
  n<-length(x_i)
  x_i<-c(5, 25, 31, 32, 34, 35, 38, 39, 39, 40, 42, 43, 43,
    43, 44, 44, 47, 47, 48, 49, 49, 49, 51, 54, 55, 55,
    55, 56, 56, 56, 58, 59, 59, 59, 59, 59, 63, 63, 64,
    64, 65, 65, 65, 66, 66, 66, 66, 66, 67, 67, 67, 68,
    69, 69, 69, 69, 71, 71, 72, 73, 73, 73, 74, 74, 76,
    76, 77, 77, 77, 77, 77, 77, 79, 79, 80, 81, 83, 83,
```


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```
        84, 86, 86, 87, 90, 91, 92, 92, 92, 92, 93, 94, 97,
        98, 98, 99, 101, 103, 105, 109, 136, 147)
  x_i
  logl<-n*log(gamma*rho**2/(varphi*(rho + varphi**2))) +
    sum(log((1 -((rho*varphi**2)*x_i + varphi*(rho + varphi**2))*exp(-
rho*x_i/varphi)/(varphi*(rho + varphi**2)))**(gamma - 1)*(varphi*x_i + 1)*exp(-
rho*x_i/varphi)))
  return(-logl)      }
janardan_re.est<-function(x_i){


  data.params<-optim(par = runif(3), janardan_re.fit, method = "SANN",
                  x_i = data, hessian = T)
  estimates<-data.params
  return(estimates)      }
janardan_re.est(x_i)
#### J MLE Only ###
jan_re.fit<-function(params, x_i){
  varphi<-params[1]
  rho<-params[2]
  n<-length(x_i)
  x_i<-c(5, 25, 31, 32, 34, 35, 38, 39, 39, 40, 42, 43, 43,
        43, 44, 44, 47, 47, 48, 49, 49, 49, 51, 54, 55, 55,
        55, 56, 56, 56, 58, 59, 59, 59, 59, 59, 63, 63, 64,
        64, 65, 65, 65, 66, 66, 66, 66, 66, 67, 67, 67, 68,
        69, 69, 69, 69, 71, 71, 72, 73, 73, 73, 74, 74, 76,
        76, 77, 77, 77, 77, 77, 77, 79, 79, 80, 81, 83, 83,
        84, 86, 86, 87, 90, 91, 92, 92, 92, 92, 93, 94, 97,
        98, 98, 99, 101, 103, 105, 109, 136, 147)
```


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```
  logl<-n*log(1*rho**2/(varphi*(rho + varphi**2))) +

    sum(log((1 -((rho*varphi**2)*x_i + varphi*(rho + varphi**2))*exp(-
rho*x_i/varphi)/(varphi*(rho + varphi**2)))**(1 - 1)*(varphi*x_i + 1)*exp(-
rho*x_i/varphi)))

  return(-logl)    }

jan_re.est<-function(x_i) {

  data.params<-optim(par = runif(2), jan_re.fit, method = "SANN",

                   x_i = data, hessian = T)

  estimates<-data.params

  return(estimates)

}

jan_re.est(x_i)

jan_re.fit<-function(params, x_i){


  varphi<-params[1]

  rho<-params[2]


  n<-length(x_i)

  x_i<-c(5, 25, 31, 32, 34, 35, 38, 39, 39, 40, 42, 43, 43,

       43, 44, 44, 47, 47, 48, 49, 49, 49, 51, 54, 55, 55,

       55, 56, 56, 56, 58, 59, 59, 59, 59, 59, 63, 63, 64,

       64, 65, 65, 65, 66, 66, 66, 66, 66, 67, 67, 67, 68,

       69, 69, 69, 69, 71, 71, 72, 73, 73, 73, 74, 74, 76,

       76, 77, 77, 77, 77, 77, 77, 79, 79, 80, 81, 83, 83,

       84, 86, 86, 87, 90, 91, 92, 92, 92, 92, 93, 94, 97,

       98, 98, 99, 101, 103, 105, 109, 136, 147)

  x_i
```


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```
  logl<-n*log(1*rho**2/(varphi*(rho + varphi**2))) +
    sum(log((1 -((rho*varphi**2)*x_i + varphi*(rho + varphi**2))*exp(-rho*x_i/varphi)/(varphi*(rho + varphi**2)))**(1 - 1)*(varphi*x_i + 1)*exp(-rho*x_i/varphi)))
  return(-logl)
}
janardan_re.est<-function(x_i){
  data.params<-optim(par = runif(2), jan_re.fit, method = "SANN",
                     x_i = data, hessian = T)
  estimates<-data.params
  return(estimates)
}
janardan_re.est(x_i)
```

Useful codes for KJ

```
  #####  KUMARASWAMY JARNADAN CDF ##############

  KJ_CDF=1-(1-(J_CDF^a))^b

  #####  KUMARASWAMY JARNADAN PDF ##############

  KJ_PDF=a*b*J_PDF*J_CDF^(a-1)*(1-J_CDF)^(b-1)
```


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```
  return(KJ_PDF)

}

windows(width=20,height=10)

par(mfrow=c(1,1))

curve(KJ_PDF_PLOT(x,5,3,2.2,5.3),0,10,col="blue",ylab=expression(paste('f',"(x)")),yli
m=c(0,0.1),xlab="x",lty=1,lwd=1)

curve(KJ_PDF_PLOT(x,3.8,3.0,0.5,1.5),0,10,col="green",add=TRUE,lty=1,lwd=1)

curve(KJ_PDF_PLOT(x,6,4,1,1.9),0,10,col="red",add=TRUE,lty=1,lwd=1)

legend("center",inset=c(0.05),cex=1.0,legend=c(expression(paste(alpha,"=",5,",",~beta,"=
",3,",",~varphi,"=",2.2,",",~rho,"=",5.3)),expression(paste(alpha,"=",3.8,",",~beta,"=",3.0,"
,",~phi,"=",0.5,",",~rho,"=",1.5)),
expression(paste(alpha,"=",6,",",~beta,"=",4,",",~phi,"=",1,",",~rho,"=",1.9))),lty=1,lwd=2
,col=c("blue","green","red"))

title(main="Figure 1: Behaviour of pdf of KJ for some parameters")


windows(width=20,height=10)

par(mfrow=c(1,1))

curve(KJ_PDF_PLOT(x,3,3.2,1.9,2),0,10,col="blue",ylab=expression(paste('f',"(x)")),yli
m=c(0,0.1),xlab="x",lty=1,lwd=1)

curve(KJ_PDF_PLOT(x,3.5,3,1.3,2.1),0,10,col="green",add=TRUE,lty=1,lwd=1)

curve(KJ_PDF_PLOT(x,4.2,3.2,3.8,4),0,10,col="red",add=TRUE,lty=1,lwd=1)

legend("center",inset=c(0.05),cex=0.75,legend=c(expression(paste(alpha,"=",3,",",~beta,"
=",3.2,",",~varphi,"=",1.9,",",~rho,"=",2)),expression(paste(alpha,"=",3.5,",",~beta,"=",3,",
",~phi,"=",1.3,",",~rho,"=",2.1)),
expression(paste(alpha,"=",4.2,",",~beta,"=",3.2,",",~phi,"=",3.8,",",~rho,"=",4))),lty=1,lw
d=2,col=c("blue","green","red"))

title(main="Figure 4.7: Behaviour of pdf of KJ for some parameters")


windows(width=20,height=10)

par(mfrow=c(1,1))
```


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```
curve(KJ_PDF_PLOT(x,3,3.2,1.9,2),0,10,col="blue",ylab=expression(paste('f',"(x)")),ylim=c(0,0.1),xlab="x",lty=1,lwd=1)

curve(KJ_PDF_PLOT(x,3.5,3,1.3,2.1),0,10,col="green",add=TRUE,lty=1,lwd=1)

curve(KJ_PDF_PLOT(x,4.2,3.2,3.8,4),0,10,col="red",add=TRUE,lty=1,lwd=1)

legend("center",inset=c(0.05),cex=0.75,legend=c(expression(paste(alpha,"=",3,",",~beta,"=",3.2,",",~varphi,"=",1.9,",",~rho,"=",2)),expression(paste(alpha,"=",3.5,",",~beta,"=",3,",",~phi,"=",1.3,",",~rho,"=",2.1)),
expression(paste(alpha,"=",4.2,",",~beta,"=",3.2,",",~phi,"=",3.8,",",~rho,"=",4))),lty=1,lwd=2,col=c("blue","green","red"))




##### modification of legend  rough work####


windows(width=20,height=10)

par(mfrow=c(1,1))

curve(KJ_PDF_PLOT(x,5,3,2.2,5.3),0,10,col="blue",ylab=expression(paste('f',"(x)")),ylim=c(0,0.1),xlab="x",lty=1,lwd=1)

curve(KJ_PDF_PLOT(x,3.8,3.0,0.5,1.5),0,10,col="green",add=TRUE,lty=1,lwd=1)

curve(KJ_PDF_PLOT(x,6,4,1,1.9),0,10,col="red",add=TRUE,lty=1,lwd=1)

legend("center",inset=c(0.05),cex=1.0,legend=c(expression(paste(alpha,"=",5,",",~beta,"=",3,",",~varphi,"=",2.2,",",~rho,"=",5.3)),expression(paste(alpha,"=",3.8,",",~beta,"=",3.0,",",~varphi,"=",0.5,",",~rho,"=",1.5)),
expression(paste(alpha,"=",6,",",~beta,"=",4,",",~varphi,"=",1,",",~rho,"=",1.9))),lty=1,lwd=2,col=c("blue","green","red"))

title(main="Figure 1: Behaviour of pdf of KJ for some parameters")


windows(width=20,height=10)

par(mfrow=c(1,1))
```


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```
curve(KJ_PDF_PLOT(x,3,3.2,1.9,2),0,10,col="blue",ylab=expression(paste('f',"(x)")),ylim=c(0,0.1),xlab="x",lty=1,lwd=1)

curve(KJ_PDF_PLOT(x,3.5,3,1.3,2.1),0,10,col="green",add=TRUE,lty=1,lwd=1)

curve(KJ_PDF_PLOT(x,4.2,3.2,3.8,4),0,10,col="red",add=TRUE,lty=1,lwd=1)

legend("center",inset=c(0.05),cex=0.75,legend=c(expression(paste(alpha,"=",3,",",~beta,"=",3.2,",",~varphi,"=",1.9,",",~rho,"=",2)),expression(paste(alpha,"=",3.5,",",~beta,"=",3,",",~varphi,"=",1.3,",",~rho,"=",2.1)),
expression(paste(alpha,"=",4.2,",",~beta,"=",3.2,",",~varphi,"=",3.8,",",~rho,"=",4))),lty=1,lwd=2,col=c("blue","green","red"))

title(main="Figure 4.7: Behaviour of pdf of KJ for some parameters")


################## EFFECT OF ALPHA ON PDF ##########


windows(width=20,height=10)

par(mfrow=c(1,1))

curve(KJ_PDF_PLOT(x,5,3,2.2,5.3),0,10,col="blue",ylab=expression(paste('f',"(x)")),ylim=c(0,0.1),xlab="x",lty=1,lwd=1)

curve(KJ_PDF_PLOT(x,3,3,2.2,5.3),0,10,col="green",add=TRUE,lty=1,lwd=1)

curve(KJ_PDF_PLOT(x,6,3,2.2,5.3),0,10,col="red",add=TRUE,lty=1,lwd=1)

legend("center",inset=c(0.05),cex=1.0,legend=c(expression(paste(alpha,"=",5,",",~beta,"=",3,",",~varphi,"=",2.2,",",~rho,"=",5.3)),expression(paste(alpha,"=",3,",",~beta,"=",3,",",~varphi,"=",2.2,",",~rho,"=",5.3)),
expression(paste(alpha,"=",6,",",~beta,"=",3,",",~varphi,"=",2.2,",",~rho,"=",5.3))),lty=1,lwd=2,col=c("blue","green","red"))

title(main="Figure 4.8: Behaviour of pdf of KJ with varying alpha value")


windows(width=20,height=10)

par(mfrow=c(1,1))

curve(KJ_PDF_PLOT(x,5,3,2.2,5.3),0,10,col="blue",ylab=expression(paste('f',"(x)")),ylim=c(0,0.1),xlab="x",lty=1,lwd=1)

curve(KJ_PDF_PLOT(x,3,3,2.2,5.3),0,10,col="green",add=TRUE,lty=1,lwd=1)
```


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```
curve(KJ_PDF_PLOT(x,6,3,2.2,5.3),0,10,col="red",add=TRUE,lty=1,lwd=1)

legend("center",inset=c(0.05),cex=1.0,legend=c(expression(paste(alpha,"=",5,",",~beta,"=
",3,",",~varphi,"=",2.2,",",~rho,"=",5.3)),expression(paste(alpha,"=",3,",",~beta,"=",3,",",~
varphi,"=",2.2,",",~rho,"=",5.3)),
expression(paste(alpha,"=",6,",",~beta,"=",3,",",~varphi,"=",2.2,",",~rho,"=",5.3))),lty=1,l
wd=2,col=c("blue","green","red"))



################# EFFECT OF BETA ON THE PDF###


windows(width=20,height=10)

par(mfrow=c(1,1))

curve(KJ_PDF_PLOT(x,5,3,2.2,5.3),0,10,col="blue",ylab=expression(paste('f',"(x)")),yli
m=c(0,0.3),xlab="x",lty=1,lwd=1)

curve(KJ_PDF_PLOT(x,5,2,2.2,5.3),0,10,col="green",add=TRUE,lty=1,lwd=1)

curve(KJ_PDF_PLOT(x,5,4,2.2,5.3),0,10,col="red",add=TRUE,lty=1,lwd=1)

legend("center",inset=c(0.05),cex=1.0,legend=c(expression(paste(alpha,"=",5,",",~beta,"=
",3,",",~varphi,"=",2.2,",",~rho,"=",5.3)),

expression(paste(alpha,"=",5,",",~beta,"=",2,",",~varphi,"=",2.2,",",~rho,"=",5.3)),

expression(paste(alpha,"=",5,",",~beta,"=",4,",",~varphi,"=",2.2,",",~rho,"=",5.3))),lty=1,l
wd=2,col=c("blue","green","red"))

title(main="Figure 4.9: Behaviour of pdf of KJ with varying beta value")



windows(width=20,height=10)

par(mfrow=c(1,1))

curve(KJ_PDF_PLOT(x,5,3,2.2,5.3),0,10,col="blue",ylab=expression(paste('f',"(x)")),yli
m=c(0,0.3),xlab="x",lty=1,lwd=1)
```


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```
curve(KJ_PDF_PLOT(x,5,2,2.2,5.3),0,10,col="green",add=TRUE,lty=1,lwd=1)

curve(KJ_PDF_PLOT(x,5,4,2.2,5.3),0,10,col="red",add=TRUE,lty=1,lwd=1)

legend("center",inset=c(0.05),cex=1.0,legend=c(expression(paste(alpha,"=",5,",",~beta,"=
",3,",",~varphi,"=",2.2,",",~rho,"=",5.3)),

expression(paste(alpha,"=",5,",",~beta,"=",2,",",~varphi,"=",2.2,",",~rho,"=",5.3)),

expression(paste(alpha,"=",5,",",~beta,"=",4,",",~varphi,"=",2.2,",",~rho,"=",5.3))),lty=1,l
wd=2,col=c("blue","green","red"))

################# EFFECT OF VARPHI ON PDF######

windows(width=20,height=10)

par(mfrow=c(1,1))

curve(KJ_PDF_PLOT(x,5,3,2.2,5.3),0,10,col="blue",ylab=expression(paste('f',"(x)")),yli
m=c(0,0.1),xlab="x",lty=1,lwd=1)

curve(KJ_PDF_PLOT(x,5,3,2,5.3),0,10,col="green",add=TRUE,lty=1,lwd=1)

curve(KJ_PDF_PLOT(x,5,3,1,5.3),0,10,col="red",add=TRUE,lty=1,lwd=1)

legend("center",inset=c(0.05),cex=1.0,legend=c(expression(paste(alpha,"=",5,",",~beta,"=
",3,",",~varphi,"=",2.2,",",~rho,"=",5.3)),

expression(paste(alpha,"=",5,",",~beta,"=",3,",",~varphi,"=",2,",",~rho,"=",5.3)),

expression(paste(alpha,"=",5,",",~beta,"=",3,",",~varphi,"=",1,",",~rho,"=",5.3))),lty=1,lw
d=2,col=c("blue","green","red"))

title(main="Figure 4.10: Behaviour of pdf of KJ with varying varphi value")
```


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```
windows(width=20,height=10)

par(mfrow=c(1,1))

curve(KJ_PDF_PLOT(x,5,3,2.2,5.3),0,10,col="blue",ylab=expression(paste('f',"(x)")),yli
m=c(0,0.1),xlab="x",lty=1,lwd=1)

curve(KJ_PDF_PLOT(x,5,3,2,5.3),0,10,col="green",add=TRUE,lty=1,lwd=1)

curve(KJ_PDF_PLOT(x,5,3,1,5.3),0,10,col="red",add=TRUE,lty=1,lwd=1)

legend("center",inset=c(0.05),cex=1.0,legend=c(expression(paste(alpha,"=",5,",",~beta,"=
",3,",",~varphi,"=",2.2,",",~rho,"=",5.3)),

expression(paste(alpha,"=",5,",",~beta,"=",3,",",~varphi,"=",2,",",~rho,"=",5.3)),

expression(paste(alpha,"=",5,",",~beta,"=",3,",",~varphi,"=",1,",",~rho,"=",5.3))),lty=1,lw
d=2,col=c("blue","green","red"))

################# EFFECT OF RHO ON PDF####

windows(width=20,height=10)

par(mfrow=c(1,1))

curve(KJ_PDF_PLOT(x,5,3,2.2,5.3),0,10,col="blue",ylab=expression(paste('f',"(x)")),yli
m=c(0,0.1),xlab="x",lty=1,lwd=1)

curve(KJ_PDF_PLOT(x,5,3,2.2,6),0,10,col="green",add=TRUE,lty=1,lwd=1)

curve(KJ_PDF_PLOT(x,5,3,2.2,7),0,10,col="red",add=TRUE,lty=1,lwd=1)

legend("center",inset=c(0.05),cex=1.0,

legend=c(expression(paste(alpha,"=",5,",",~beta,"=",3,",",~varphi,"=",2.2,",",~rho,"=",5.3)
),

          expression(paste(alpha,"=",5,",",~beta,"=",2,",",~varphi,"=",2.2,",",~rho,"=",6)),
```


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```
expression(paste(alpha,"=",5,",",~beta,"=",4,",",~varphi,"=",2.2,",",~rho,"=",7))),lty=1,lwd=2,col=c("blue","green","red"))

title(main="Figure 4.11: Behaviour of pdf of KJ with varying rho value")




windows(width=20,height=10)

par(mfrow=c(1,1))

curve(KJ_PDF_PLOT(x,5,3,2.2,5.3),0,10,col="blue",ylab=expression(paste('f',"(x)")),ylim=c(0,0.1),xlab="x",lty=1,lwd=1)

curve(KJ_PDF_PLOT(x,5,3,2.2,6),0,10,col="green",add=TRUE,lty=1,lwd=1)

curve(KJ_PDF_PLOT(x,5,3,2.2,7),0,10,col="red",add=TRUE,lty=1,lwd=1)

legend("center",inset=c(0.05),cex=1.0,

legend=c(expression(paste(alpha,"=",5,",",~beta,"=",3,",",~varphi,"=",2.2,",",~rho,"=",5.3)),

        expression(paste(alpha,"=",5,",",~beta,"=",2,",",~varphi,"=",2.2,",",~rho,"=",6)),

expression(paste(alpha,"=",5,",",~beta,"=",4,",",~varphi,"=",2.2,",",~rho,"=",7))),lty=1,lwd=2,col=c("blue","green","red"))
```

Useful codes for EKJ

```
################ EKJ PDF PLOT ##############################

EKJ_PDF_PLOT<-function(x, a, b, f,r,g){

 ######   JARNADAN CDF  ###############

 A=f*(r-f^2)

 B=exp(-(r/f)*x)
```


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```
J_CDF=1-(((A+(r*(f^2)*x))/(f*(r+f^2)))*B)

######  JARNADAN PDF  ##############

C=f*(r+f^2)

B=exp(-(r/f)*x)

J_PDF=((f^2)/C)*(1+(r*x))*B

#####  KUMARASWAMY JARNADAN CDF ##############

KJ_CDF=1-(1-(J_CDF^a))^b

#####  KUMARASWAMY JARNADAN PDF ##############

KJ_PDF=a*b*J_PDF*J_CDF^(a-1)*(1-J_CDF)^(b-1)

######  EXPONENTIATED KUMARASWAMY JARNADAN CDF #########

EKJ_CDF=(KJ_CDF)^g

######  EXPONENTIATED KUMARASWAMY JARNADAN PDF #########

EKJ_PDF=g*(KJ_PDF)*(KJ_CDF)^(g-1)

EKJ_PDF

return(EKJ_PDF)

}

windows(width=20,height=10)

par(mfrow=c(1,1))

curve(EKJ_PDF_PLOT(x,5,3,2.2,5.3,1.8),0,10,col="blue",ylab=expression(paste('f',"(x)")),ylim=c(0,0.1),xlab="x",lty=1,lwd=1)

curve(EKJ_PDF_PLOT(x,3.8,3.0,0.5,1.5,2.5),0,10,col="green",add=TRUE,lty=1,lwd=1)

curve(EKJ_PDF_PLOT(x,6,4,1,1.9,5),0,10,col="red",add=TRUE,lty=1,lwd=1)

legend("center",inset=c(0.05),cex=1.0,legend=c(expression(paste(alpha,"=",5,",",~beta,"=",3,",",~phi,"=",2.2,",",~rho,"=",5.3,",",~gamma,"=",1.8)),expression(paste(alpha,"=",3.8,",",~beta,"=",3.0,",",~phi,"=",0.5,",",~rho,"=",1.5,",",~gamma,"=",2.5)),
expression(paste(alpha,"=",6,",",~beta,"=",4,",",~phi,"=",1,",",~rho,"=",1.9,",",~gamma,"=",5))),lty=1,lwd=2,col=c("blue","green","red"))

title(main="Figure 4.1.1a: Behaviour of pdf of EKJ for some parameters")
```


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```
windows(width=20,height=10)

par(mfrow=c(1,1))

curve(EKJ_PDF_PLOT(x,3,3.2,1.9,2,3.1),0,10,col="blue",ylab=expression(paste('f',"(x)")
),ylim=c(0,0.1),xlab="x",lty=1,lwd=1)

curve(EKJ_PDF_PLOT(x,3.5,3,1.3,2.1,0.4),0,10,col="green",add=TRUE,lty=1,lwd=1)

curve(EKJ_PDF_PLOT(x,4.2,3.2,3.8,4,1.8),0,10,col="red",add=TRUE,lty=1,lwd=1)

legend("center",inset=c(0.05),cex=0.8,legend=c(expression(paste(alpha,"=",3,",",~beta,"=
",3.2,",",~phi,"=",1.9,",",~rho,"=",2,",",~gamma,"=",3.1)),expression(paste(alpha,"=",3.5,"
,",~beta,"=",3,",",~phi,"=",1.3,",",~rho,"=",2.1,",",~gamma,"=",.4)),
expression(paste(alpha,"=",4.2,",",~beta,"=",3.2,",",~phi,"=",3.8,",",~rho,"=",4,",",~gamm
a,"=",1.8))),lty=1,lwd=2,col=c("blue","green","red"))

title(main="Figure 4.1.1b: Behaviour of pdf of EKJ for some parameters")


windows(width=20,height=10)

par(mfrow=c(1,1))

curve(EKJ_PDF_PLOT(x,0.2,1.5,2,4,2),0,10,col="blue",ylab=expression(paste('f',"(x)")),
ylim=c(0,0.1),xlab="x",lty=1,lwd=1)

curve(EKJ_PDF_PLOT(x,7,2.5,0.5,2,1.3),0,10,col="green",add=TRUE,lty=1,lwd=1)

curve(EKJ_PDF_PLOT(x,7,4.2,3.3,4.5,1.1),0,10,col="red",add=TRUE,lty=1,lwd=1)

legend("center",inset=c(0.05),cex=1.0,legend=c(expression(paste(alpha,"=",.2,",",~beta,"=
",1.5,",",~phi,"=",2,",",~rho,"=",4,",",~gamma,"=",2)),expression(paste(alpha,"=",7,",",~b
eta,"=",2.5,",",~phi,"=",.5,",",~rho,"=",2,",",~gamma,"=",1.3)),
expression(paste(alpha,"=",7,",",~beta,"=",4.2,",",~phi,"=",3.3,",",~rho,"=",4.5,",",~gamm
a,"=",1.1))),lty=1,lwd=2,col=c("blue","green","red"))

title(main="Figure 4.1.1c: Behaviour of pdf of EKJ for some parameters")


windows(width=20,height=10)

par(mfrow=c(1,1))

curve(EKJ_PDF_PLOT(x,2,1.5,0.3,3,2),0,10,col="blue",ylab=expression(paste('f',"(x)")),
ylim=c(0,0.1),xlab="x",lty=1,lwd=1)

curve(EKJ_PDF_PLOT(x,3,3.8,1.5,1.5,2.5),0,10,col="green",add=TRUE,lty=1,lwd=1)
```


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```
curve(EKJ_PDF_PLOT(x,8,4,3,5,1),0,10,col="red",add=TRUE,lty=1,lwd=1)

legend("center",inset=c(0.1),cex=0.75,legend=c(expression(paste(alpha,"=",2,",",~beta,"=
",1.5,",",~phi,"=",0.3,",",~rho,"=",3,",",~gamma,"=",2)),expression(paste(alpha,"=",3,",",~
beta,"=",3.8,",",~phi,"=",1.5,",",~rho,"=",1.5,",",~gamma,"=",2.5)),
expression(paste(alpha,"=",8,",",~beta,"=",4,",",~phi,"=",3,",",~rho,"=",5,",",~gamma,"=",
1))),lty=1,lwd=2,col=c("blue","green","red"))

title(main="Figure 4.1.1d: Behaviour of pdf of EKJ for some parameters")


windows(width=20,height=10)

par(mfrow=c(1,1))

curve(EKJ_PDF_PLOT(x,3,3.2,1.9,2,3.1),0,10,col="blue",ylab=expression(paste('f',"(x)")
),ylim=c(0,0.1),xlab="x",lty=1,lwd=1)

curve(EKJ_PDF_PLOT(x,3.5,3,1.3,2.1,0.4),0,10,col="green",add=TRUE,lty=1,lwd=1)

curve(EKJ_PDF_PLOT(x,4.2,3.2,3.8,4,1.8),0,10,col="red",add=TRUE,lty=1,lwd=1)

legend("center",inset=c(0.05),cex=0.8,legend=c(expression(paste(alpha,"=",3,",",~beta,"=
",3.2,",",~phi,"=",1.9,",",~rho,"=",2,",",~gamma,"=",3.1)),expression(paste(alpha,"=",3.5,"
,",~beta,"=",3,",",~phi,"=",1.3,",",~rho,"=",2.1,",",~gamma,"=",.4)),
expression(paste(alpha,"=",4.2,",",~beta,"=",3.2,",",~phi,"=",3.8,",",~rho,"=",4,",",~gamm
a,"=",1.8))),lty=1,lwd=2,col=c("blue","green","red"))

title(main="Figure 4.12: Behaviour of pdf of EKJ for some parameters")

##################################### EFFECT OF VARYING PARAMETERS ON
EKJ ########################

windows(width=20,height=10)

par(mfrow=c(1,1))

curve(EKJ_PDF_PLOT(x,3,3.2,1.9,2,3.1),0,10,col="blue",ylab=expression(paste('f',"(x)")
),ylim=c(0,0.1),xlab="x",lty=1,lwd=1)

curve(EKJ_PDF_PLOT(x,3.5,3,1.3,2.1,0.4),0,10,col="green",add=TRUE,lty=1,lwd=1)

curve(EKJ_PDF_PLOT(x,4.2,3.2,3.8,4,1.8),0,10,col="red",add=TRUE,lty=1,lwd=1)

legend("center",inset=c(0.05),cex=0.8,legend=c(expression(paste(alpha,"=",3,",",~beta,"=
",3.2,",",~varphi,"=",1.9,",",~rho,"=",2,",",~gamma,"=",3.1)),expression(paste(alpha,"=",3
.5,",",~beta,"=",3,",",~varphi,"=",1.3,",",~rho,"=",2.1,",",~gamma,"=",.4)),
expression(paste(alpha,"=",4.2,",",~beta,"=",3.2,",",~varphi,"=",3.8,",",~rho,"=",4,",",~ga
mma,"=",1.8))),lty=1,lwd=2,col=c("blue","green","red"))
```


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```
title(main="Figure 4.12: Behaviour of pdf of EKJ for some parameters")


## EFFECT OF VARYING PARAMETERS ON EKJ ####

windows(width=20,height=10)

par(mfrow=c(1,1))

curve(EKJ_PDF_PLOT(x,3,3.2,1.9,2,3.1),0,10,col="blue",ylab=expression(paste('f',"(x)")
),ylim=c(0,0.1),xlab="x",lty=1,lwd=1)

curve(EKJ_PDF_PLOT(x,3.5,3,1.3,2.1,0.4),0,10,col="green",add=TRUE,lty=1,lwd=1)

curve(EKJ_PDF_PLOT(x,4.2,3.2,3.8,4,1.8),0,10,col="red",add=TRUE,lty=1,lwd=1)

legend("center",inset=c(0.05),cex=0.8,legend=c(expression(paste(alpha,"=",3,",",~beta,"=
",3.2,",",~varphi,"=",1.9,",",~rho,"=",2,",",~gamma,"=",3.1)),expression(paste(alpha,"=",3
.5,",",~beta,"=",3,",",~varphi,"=",1.3,",",~rho,"=",2.1,",",~gamma,"=",.4)),
expression(paste(alpha,"=",4.2,",",~beta,"=",3.2,",",~varphi,"=",3.8,",",~rho,"=",4,",",~ga
mma,"=",1.8))),lty=1,lwd=2,col=c("blue","green","red"))

title(main="Figure 4.12: Behaviour of pdf of EKJ for some parameters")


########### EFFECT OF VARYING ALPHA ON EKJ #########

windows(width=20,height=10)

par(mfrow=c(1,1))

curve(EKJ_PDF_PLOT(x,3,3.2,1.9,2,3.1),0,10,col="blue",ylab=expression(paste('f',"(x)")
),ylim=c(0,0.1),xlab="x",lty=1,lwd=1)

curve(EKJ_PDF_PLOT(x,4.5,3.2,1.9,2,3.1),0,10,col="green",add=TRUE,lty=1,lwd=1)

curve(EKJ_PDF_PLOT(x,4.2,3.2,1.9,2,3.1),0,10,col="red",add=TRUE,lty=1,lwd=1)

legend("center",inset=c(0.05),cex=0.8,legend=c(expression(paste(alpha,"=",3,",",~beta,"=
",3.2,",",~varphi,"=",1.9,",",~rho,"=",2,",",~gamma,"=",3.1)),expression(paste(alpha,"=",3
.5,",",~beta,"=",3.2,",",~varphi,"=",1.9,",",~rho,"=",2,",",~gamma,"=",3.1)),
expression(paste(alpha,"=",4.2,",",~beta,"=",3.2,",",~varphi,"=",1.9,",",~rho,"=",2,",",~ga
mma,"=",3.1))),lty=1,lwd=2,col=c("blue","green","red"))

title(main="Figure 4.13: Effect of alpha on behaviour of pdf of EKJ")
```


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```
windows(width=20,height=10)

par(mfrow=c(1,1))

curve(EKJ_PDF_PLOT(x,3,3.2,1.9,2,3.1),0,10,col="blue",ylab=expression(paste('f',"(x)")),ylim=c(0,0.1),xlab="x",lty=1,lwd=1)

curve(EKJ_PDF_PLOT(x,4.5,3.2,1.9,2,3.1),0,10,col="green",add=TRUE,lty=1,lwd=1)

curve(EKJ_PDF_PLOT(x,4.2,3.2,1.9,2,3.1),0,10,col="red",add=TRUE,lty=1,lwd=1)

legend("center",inset=c(0.05),cex=0.8,legend=c(expression(paste(alpha,"=",3,",",~beta,"=",3.2,",",~varphi,"=",1.9,",",~rho,"=",2,",",~gamma,"=",3.1)),expression(paste(alpha,"=",3.5,",",~beta,"=",3.2,",",~varphi,"=",1.9,",",~rho,"=",2,",",~gamma,"=",3.1)),
expression(paste(alpha,"=",4.2,",",~beta,"=",3.2,",",~varphi,"=",1.9,",",~rho,"=",2,",",~gamma,"=",3.1))),lty=1,lwd=2,col=c("blue","green","red"))


########### EFFECT OF VARYING BETA ON EKJ #########

windows(width=20,height=10)

par(mfrow=c(1,1))

curve(EKJ_PDF_PLOT(x,3,3.2,1.9,2,3.1),0,10,col="blue",ylab=expression(paste('f',"(x)")),ylim=c(0,0.1),xlab="x",lty=1,lwd=1)

curve(EKJ_PDF_PLOT(x,3,4,1.9,2,3.1),0,10,col="green",add=TRUE,lty=1,lwd=1)

curve(EKJ_PDF_PLOT(x,3,4.5,1.9,2,3.1),0,10,col="red",add=TRUE,lty=1,lwd=1)

legend("center",inset=c(0.05),cex=0.8,legend=c(expression(paste(alpha,"=",3,",",~beta,"=",3.2,",",~varphi,"=",1.9,",",~rho,"=",2,",",~gamma,"=",3.1)),expression(paste(alpha,"=",3,",",~beta,"=",4,",",~varphi,"=",1.9,",",~rho,"=",2,",",~gamma,"=",3.1)),
expression(paste(alpha,"=",3,",",~beta,"=",4.5,",",~varphi,"=",1.9,",",~rho,"=",2,",",~gamma,"=",3.1))),lty=1,lwd=2,col=c("blue","green","red"))

title(main="Figure 4.14: Effect of beta on behaviour of pdf of EKJ")


windows(width=20,height=10)

par(mfrow=c(1,1))
```


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```
curve(EKJ_PDF_PLOT(x,3,3.2,1.9,2,3.1),0,10,col="blue",ylab=expression(paste('f',"(x)")
),ylim=c(0,0.1),xlab="x",lty=1,lwd=1)

curve(EKJ_PDF_PLOT(x,3,4,1.9,2,3.1),0,10,col="green",add=TRUE,lty=1,lwd=1)

curve(EKJ_PDF_PLOT(x,3,4.5,1.9,2,3.1),0,10,col="red",add=TRUE,lty=1,lwd=1)

legend("center",inset=c(0.05),cex=0.8,legend=c(expression(paste(alpha,"=",3,",",~beta,"=
",3.2,",",~varphi,"=",1.9,",",~rho,"=",2,",",~gamma,"=",3.1)),expression(paste(alpha,"=",3
,",",~beta,"=",4,",",~varphi,"=",1.9,",",~rho,"=",2,",",~gamma,"=",3.1)),
expression(paste(alpha,"=",3,",",~beta,"=",4.5,",",~varphi,"=",1.9,",",~rho,"=",2,",",~gam
ma,"=",3.1))),lty=1,lwd=2,col=c("blue","green","red"))


########### EFFECT OF VARYING VARPHI ON EKJ #########

windows(width=20,height=10)

par(mfrow=c(1,1))

curve(EKJ_PDF_PLOT(x,3,3.2,1.9,2,3.1),0,10,col="blue",ylab=expression(paste('f',"(x)")
),ylim=c(0,0.1),xlab="x",lty=1,lwd=1)

curve(EKJ_PDF_PLOT(x,3,3.2,2,2,3.1),0,10,col="green",add=TRUE,lty=1,lwd=1)

curve(EKJ_PDF_PLOT(x,3,3.2,2.5,2,3.1),0,10,col="red",add=TRUE,lty=1,lwd=1)

legend("center",inset=c(0.05),cex=0.8,legend=c(expression(paste(alpha,"=",3,",",~beta,"=
",3.2,",",~varphi,"=",1.9,",",~rho,"=",2,",",~gamma,"=",3.1)),expression(paste(alpha,"=",3
,",",~beta,"=",4,",",~varphi,"=",2,",",~rho,"=",2,",",~gamma,"=",3.1)),
expression(paste(alpha,"=",3,",",~beta,"=",4.5,",",~varphi,"=",2.5,",",~rho,"=",2,",",~gam
ma,"=",3.1))),lty=1,lwd=2,col=c("blue","green","red"))

title(main="Figure 4.15: Effect of Varphi on behaviour of pdf of EKJ")



windows(width=20,height=10)

par(mfrow=c(1,1))

curve(EKJ_PDF_PLOT(x,3,3.2,1.9,2,3.1),0,10,col="blue",ylab=expression(paste('f',"(x)")
),ylim=c(0,0.1),xlab="x",lty=1,lwd=1)

curve(EKJ_PDF_PLOT(x,3,3.2,2,2,3.1),0,10,col="green",add=TRUE,lty=1,lwd=1)

curve(EKJ_PDF_PLOT(x,3,3.2,2.5,2,3.1),0,10,col="red",add=TRUE,lty=1,lwd=1)
```


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```
legend("center",inset=c(0.05),cex=0.8,legend=c(expression(paste(alpha,"=",3,",",~beta,"=",3.2,",",~varphi,"=",1.9,",",~rho,"=",2,",",~gamma,"=",3.1)),expression(paste(alpha,"=",3,",",~beta,"=",4,",",~varphi,"=",2,",",~rho,"=",2,",",~gamma,"=",3.1)),
expression(paste(alpha,"=",3,",",~beta,"=",4.5,",",~varphi,"=",2.5,",",~rho,"=",2,",",~gamma,"=",3.1))),lty=1,lwd=2,col=c("blue","green","red"))


########### EFFECT OF VARYING RHO ON EKJ #########

windows(width=20,height=10)

par(mfrow=c(1,1))

curve(EKJ_PDF_PLOT(x,3,3.2,1.9,2,3.1),0,10,col="blue",ylab=expression(paste('f',"(x)")),ylim=c(0,0.1),xlab="x",lty=1,lwd=1)

curve(EKJ_PDF_PLOT(x,3,3.2,1.9,3,3.1),0,10,col="green",add=TRUE,lty=1,lwd=1)

curve(EKJ_PDF_PLOT(x,3,3.2,1.9,1.6,3.1),0,10,col="red",add=TRUE,lty=1,lwd=1)

legend("center",inset=c(0.05),cex=0.8,legend=c(expression(paste(alpha,"=",3,",",~beta,"=",3.2,",",~varphi,"=",1.9,",",~rho,"=",2,",",~gamma,"=",3.1)),expression(paste(alpha,"=",3,",",~beta,"=",4,",",~varphi,"=",2,",",~rho,"=",3,",",~gamma,"=",3.1)),
expression(paste(alpha,"=",3,",",~beta,"=",4.5,",",~varphi,"=",2.5,",",~rho,"=",1.6,",",~gamma,"=",3.1))),lty=1,lwd=2,col=c("blue","green","red"))

title(main="Figure 4.16: Effect of Rho on behaviour of pdf of EKJ")


windows(width=20,height=10)

par(mfrow=c(1,1))

curve(EKJ_PDF_PLOT(x,3,3.2,1.9,2,3.1),0,10,col="blue",ylab=expression(paste('f',"(x)")),ylim=c(0,0.1),xlab="x",lty=1,lwd=1)

curve(EKJ_PDF_PLOT(x,3,3.2,1.9,3,3.1),0,10,col="green",add=TRUE,lty=1,lwd=1)

curve(EKJ_PDF_PLOT(x,3,3.2,1.9,1.6,3.1),0,10,col="red",add=TRUE,lty=1,lwd=1)

legend("center",inset=c(0.05),cex=0.8,legend=c(expression(paste(alpha,"=",3,",",~beta,"=",3.2,",",~varphi,"=",1.9,",",~rho,"=",2,",",~gamma,"=",3.1)),expression(paste(alpha,"=",3,",",~beta,"=",4,",",~varphi,"=",2,",",~rho,"=",3,",",~gamma,"=",3.1)),
expression(paste(alpha,"=",3,",",~beta,"=",4.5,",",~varphi,"=",2.5,",",~rho,"=",1.6,",",~gamma,"=",3.1))),lty=1,lwd=2,col=c("blue","green","red"))

########### EFFECT OF GAMMA ON EKJ #########
```


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```
windows(width=20,height=10)

par(mfrow=c(1,1))

curve(EKJ_PDF_PLOT(x,3,3.2,1.9,2,3.1),0,10,col="blue",ylab=expression(paste('f',"(x)")
),ylim=c(0,0.15),xlab="x",lty=1,lwd=1)

curve(EKJ_PDF_PLOT(x,3,3.2,1.9,2,4.1),0,10,col="green",add=TRUE,lty=1,lwd=1)

curve(EKJ_PDF_PLOT(x,3,3.2,1.9,2,5.1),0,10,col="red",add=TRUE,lty=1,lwd=1)

legend("center",inset=c(0.05),cex=0.8,legend=c(expression(paste(alpha,"=",3,",",~beta,"=
",3.2,",",~varphi,"=",1.9,",",~rho,"=",2,",",~gamma,"=",3.1)),expression(paste(alpha,"=",3
,",",~beta,"=",4,",",~varphi,"=",2,",",~rho,"=",3,",",~gamma,"=",4.1)),
expression(paste(alpha,"=",3,",",~beta,"=",4.5,",",~varphi,"=",2.5,",",~rho,"=",1.6,",",~ga
mma,"=",5.1))),lty=1,lwd=2,col=c("blue","green","red"))

title(main="Figure 4.17: Effect of gamma on behaviour of pdf of EKJ"

windows(width=20,height=10)

par(mfrow=c(1,1))

curve(EKJ_PDF_PLOT(x,3,3.2,1.9,2,3.1),0,10,col="blue",ylab=expression(paste('f',"(x)")
),ylim=c(0,0.15),xlab="x",lty=1,lwd=1)

curve(EKJ_PDF_PLOT(x,3,3.2,1.9,2,4.1),0,10,col="green",add=TRUE,lty=1,lwd=1)

curve(EKJ_PDF_PLOT(x,3,3.2,1.9,2,5.1),0,10,col="red",add=TRUE,lty=1,lwd=1)

legend("center",inset=c(0.05),cex=0.8,legend=c(expression(paste(alpha,"=",3,",",~beta,"=
",3.2,",",~varphi,"=",1.9,",",~rho,"=",2,",",~gamma,"=",3.1)),expression(paste(alpha,"=",3
,",",~beta,"=",4,",",~varphi,"=",2,",",~rho,"=",3,",",~gamma,"=",4.1)),
expression(paste(alpha,"=",3,",",~beta,"=",4.5,",",~varphi,"=",2.5,",",~rho,"=",1.6,",",~ga
mma,"=",5.1))),lty=1,lwd=2,col=c("blue","green","red"))
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Index

Digitized by UMaT Library

E

Digitized by UMaT Library

F

G

H

I

J

Digitized by UMaT Library

Digitized by UMaT Library

Digitized by UMaT Library

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